Calculus with Business and Economic Applications

MATHEMATICS 1431 Calculus with Business and Economic Applications. Differential and integral calculus of algebraic, logarithmic, and exponential functions; applications to business and economics, such as maximum-minimum problems, marginal analysis, and exponential growth models.

17 lessons and 2 exams. 3 hours of college credit. 08/19/10.

Prerequisite: MATH 1021 or equivalent. Credit will be given for only one of the following: MATH 1431, 1441, 1550.
Table of Contents

How to Take an Independent Learning Course ................................................................. iii
Where the Books Are ...................................................................................................... vii

Syllabus........................................................................................................................................ S–1
Textbooks
  Other Materials
Nature and Purpose of the Course
Working with the Course Material
Reading Assignments
Preparation of Lesson Assignments
  Suggested Study Techniques
  Academic Integrity
Contact Information
Examinations and Grading Policy
Transcript Information
Examination Proctors

Lesson 1: Limits ......................................................................................................................... 1–1
Lesson 2: One-Sided Limits and Continuity: The Derivative ...................... 2–1
Lesson 3: Basic Rules of Differentiation: The Product and Quotient Rules .......... 3–1
Lesson 4: The Chain Rule ....................................................................................................... 4–1
Lesson 5: Marginal Functions in Economies: Higher-Order Derivatives .......... 5–1
Lesson 6: Applications of the First Derivative ................................................................. 6–1
Lesson 7: Applications of the Second Derivative ............................................................. 7–1
Lesson 8: Curve Sketching .................................................................................................. 8–1
Lesson 9: Optimization .......................................................................................................... 9–1
Mid-Course Examination ....................................................................................................... MC–1
Lesson 10: Compound Interest ............................................................................................ 10–1
Lesson 11: Differentiation of Exponential Functions: Differentiation of Logarithmic Functions ................................................................. 11–1
Lesson 12: Exponential Functions as Mathematical Models .............................................. 12–1
Lesson 13: Antiderivatives and the Rules of Integration ...................................................... 13–1
# Table of Contents

**Lesson 14:** Integration by Substitution ................................................................. 14–1

**Lesson 15:** Area and the Definite Integral: The Fundamental Theorem of Calculus ................................................................. 15–1

**Lesson 16:** Evaluating Definite Integrals: Area between Two Curves .......... 16–1

**Lesson 17:** Applications of the Definite Integral to Business and Economics .... 17–1

**Final Examination** .................................................................................................. F–1

**Appendix A** ............................................................................................................. A–1

*College Exam Information* ...................................................................................... A–3

*Exam Proctor Information Form* ........................................................................... A–5
Welcome

Congratulations! By enrolling in this course, you have taken a major step toward achieving your educational goals. We would like to let you know what you need to do before you start studying and remind you of some of our procedures and rules (for a full listing, please check our website at www.outreach.lsu.edu/idl).

Textbooks

To find out which textbooks you need for the course, refer to the course syllabus. To order your textbooks, see “Where the Books Are” on page vii in this course guide. If you wish to order your books by mail, please use the “Textbook Order Form” that is enclosed in your packet of materials.

Other Materials

Check to see if you need any supplementary materials, or if you need to arrange any interviews or extra materials for projects. You can find this
How to Take an IDL Course

information by reading “Other Materials” section in the course syllabus, and then reviewing the Lesson Assignments at the end of each lesson.

Time Limits & Extensions

Start planning your timetable now. Please note the following rules concerning timing:

• You have an enrollment period of nine months from the date of your enrollment to complete this course. If you are an LSU student, your dean may have given you a shorter deadline. If you cannot finish your course within nine months, you can make a written request for an extension of an additional three months, provided we receive your request before your course enrollment expires. It may be possible to request a second extension. Second extensions are given when you have made progress in the course, but have encountered significant difficulty in reaching completion. For a second extension, you must make a written request, explaining your circumstances. The request must be received prior to the expiration of the first extension period. There is a fee for each extension.

• We will accept a maximum of three lessons every seven calendar days. There must be an interval of seven days between each set of three lessons. If you submit more than three lessons in a seven-day period, the additional lessons will be held until they are eligible, and then logged in and forwarded to your instructor for grading. If more than six lessons are received in a seven-day period, the ineligible lessons will be returned to you for resubmission.

• We recommend that you submit your first lesson and wait for your instructor’s feedback before submitting additional lessons. That way, you will know whether you have a clear understanding of your instructor’s expectations.

• We ask your instructor to grade your lessons and exams within two weeks, but during campus examination periods and vacation time, it may take your instructor longer to return your work.

• If you are a graduating senior, you must allow at least four weeks between taking your final exam and expecting your transcript to reach your university.

Exams & Grading

As soon as possible, begin to make arrangements for where you will take your examinations. To find out about your options, read the College
Examination Information in the appendix of this course guide. Then (if you do not plan to take your exam at LSU-BR), fill in the Exam Proctor Information Form in the appendix and send it to us before you start the course, so that we will have all your information prepared when you are ready to take your examinations.

Before we can send your exams to your exam proctor or allow you to take your exams in our office, we must have received all of your completed lesson assignments that precede the exam. Exams may not be taken until all of the assigned lessons have been submitted and accepted within our three-lessons-per-seven-days requirement. If an instructor grades any of your assignments as incomplete, you will not be eligible to take your exam(s) until you have completed the lessons.

Each course has its own grading scale, but for nearly all courses you must pass the final exam to receive credit for the course.

Remember that you only have one chance to take your examinations. You will not be allowed to repeat a failed exam within the same enrollment period. If you need to re-enroll in a course, please contact our office.

Typically, you will have three hours to take a three-credit-hour exam.

You should take your exam at least four weeks before you need your grade.

**Refunds & Transfers**

We hope you have enrolled in the course you wanted, but if not, you have 30 days to make a written request to receive an 80% refund, provided you have not submitted any lessons. Alternatively, you can transfer to another course, provided you make your written request within three months and pay a transfer fee. If you transfer, your enrollment period begins on the date of your original enrollment. Enrollments may not be transferred to another student.

If you want to withdraw from a course after the refund and transfer periods have expired, please let us know in writing that you have decided to drop the course. Provided that you do not sign in to take your final examination, there will be no record on your transcript to indicate that you ever enrolled in the course.
Electronic Resources

To assist you with your independent learning experience, we have created StudyNet, available at www.outreach.lsu.edu/idl. Once you reach the site, click “college services” from the enrolled students menu items.

This site includes up-to-date information on policies and procedures as well as resources and a number of online options to help you with your course. Using StudyNet, you may check to see whether we have received a lesson or exam, find out your grades, enroll in a course, submit change of address and exam proctor forms, and locate contact information for LSU Independent & Distance Learning staff members.

Contact Us

If you need us to clarify any of our policies, let us know! We are available by phone, by mail, by fax, and by e-mail.

- For questions regarding enrollment, lessons, or testing, call 800-234-5046.
- For questions regarding difficulty locating textbooks, call 800-234-5046 and ask to speak to the publications section.
- Our fax number is 225-578-3090.
- Our e-mail address is iServices@outreach.lsu.edu.
- Our mailing address is:
  LSU Independent & Distance Learning
  1225 Pleasant Hall
  Louisiana State University
  Baton Rouge, LA 70803-1508
General Textbook Information

You must buy your own textbooks and other supplies. The bookstores listed below stock the textbooks used in LSU Independent & Distance Learning courses. If the books are not available from one of the following bookstores, they may be available from the publisher, online vendors, or from other local booksellers.

Other required materials for your course such as calculators, binders, etc., may be purchased locally.

Secondhand and paperback copies of textbooks are often available. If secondhand or paperback books are desired, make that request at the time the order is placed.

You must use the edition of the textbook specified by the course guide!
Please do not ask if an alternate book is available. Always order using the ISBN provided in the syllabus to insure that you have the correct materials.

All of the bookstores listed below are independently owned and operated; they are not operated by Louisiana State University or LSU Independent & Distance Learning. Please be aware of refund and buy-back policies before you make your purchase.

LSU Online Bookstore

Specialty Books is the official bookstore for LSU Continuing Education. To order your textbooks online, go to www.specialty-books.com/LSU and follow the instructions provided.
Where the Books Are

**Specialty Books**  
6000 Poston Road  
Athens, OH 45701  
800-466-7132  
www.specialty-books.com/LSU

**Note:** Specialty Books is not a part of LSU; any questions or concerns should be directed to their representatives.

**Local Baton Rouge Bookstores**

The following Baton Rouge bookstores also carry course materials and textbooks:

**Chimes Textbook Exchange** (Gonzales location)  
432 N. Burnside Avenue  
Gonzales, LA 70737  
800-925-1704 (toll-free)  
E-mail: Chimestext@eatel.net

**Chimes Textbook Exchange**  
268 W. Chimes St.  
Baton Rouge, LA 70802  
225-383-5161  
www.chimestext.com

**Co-Op Bookstore**  
3960 Burbank Dr.  
Baton Rouge, LA 70808  
225-383-9870 or 866-383-9870 (toll-free)  
E-mail: books@coopbookstore.com  
www.coopbookstore.com

**Note:** Always order using the ISBN provided in the syllabus to insure that you have the correct materials. These bookstores carry a wide variety of books that are used in on-campus and well as IDL courses. Be sure to indicate that you are ordering a book for an independent study course.

**Other Online Options**

Books may also be obtained from any vendor that sells college-level textbooks, including online booksellers, university bookstores, and publishers, but you must purchase the correct edition of the textbook(s). Independent & Distance Learning does not sell textbooks (any exceptions are specifically indicated in
your course guide), so please do not send money for textbooks to Independent & Distance Learning.

**You must use the correct edition of the textbook, as specified in your course guide.** Please take care to provide the correct information about the author, title, edition, ISBN, and date of publication when ordering your books. If complete information is not given when the order is placed, the wrong edition may be sent.

The best way to make sure that you order the correct textbook is to order by the ISBN provided in the syllabus.

Syllabus

MATH 1431—Calculus with Business and Economic Applications

- Textbooks
- Nature and Purpose of the Course
- Working with the Course Material
- Reading Assignments
- Preparation of Lesson Assignments
- Contact Information
- Examinations and Grading Policy
- Transcript Information
- Examination Proctors

Textbooks


ISBN-10: 0-495-38754-1

It is recommended that you buy your textbooks as soon as possible. If you wait, you may not be able to find the correct textbook. During the nine months that you have to complete the course, a revised version of the course may be released. If the newer version of the course uses a more recent edition of the textbook or a different textbook from the one required by the version that you are enrolled in, you may have difficulty getting the textbook that you need for your version of the course. For that reason, you should buy your textbooks as soon as possible.

If you have trouble finding a book, check the list of recommended bookstores on the IDL website and order by the ISBN, not the title. If you are outside of the Baton Rouge area and try to buy your textbook locally or from an online bookstore and have difficulty locating the correct textbook or the required edition, please call one of the recommended bookstores. These bookstores try to maintain an inventory of all IDL textbooks. Be sure to specify
that you need a textbook for the Independent & Distance Learning version of the course and **verify the ISBN number** to make sure you get the correct edition of the textbook.

### Nature and Purpose of the Course

Mathematics 1431 is an introduction to Calculus and its applications to business and economics. It is assumed you have completed a college level algebra course. The material to be covered will be differential and integral calculus of algebraic, logarithmic and exponential functions. This material is usually included under the general term elementary calculus. The objective of this course is to familiarize you with the powerful tools of calculus and to enable you to apply these tools to solve business and economic problems. In addition, this course will prepare you for higher level calculus-based business and economics courses. A review of college algebra is strongly advised. This is a problem solving course. The proofs of theorems, while they may be enlightening, need not be learned.

### Learning Objectives for MATH 1431

The student should be able to do the following:

**A. Limits and Continuity**
1. Evaluate limits from a graph or from an formulaic function
2. Evaluate limits at points of continuity
3. Know what continuity implies about a graph and behavior of a function
4. Determine points of discontinuity for functions defined as formulas or graphs

**B. Differentiation**
1. Know the various interpretations of the derivative (velocity, rate of change, slope of tangent line)
2. Evaluate the derivatives of simple functions using a difference quotient
3. Find tangent lines and be able to use them as linear approximations
4. Find critical values, local extrema and the intervals of concavity for differentiable functions
5. Find absolute extrema of constrained functions
6. Solve optimization problems using calculus
7. Determine and interpret the elasticity of demand for a demand equation

**C. Integration**
1. Understand anti-derivatives and know the basic anti-derivative formulas
2. Have an understanding of the Riemann Integral as a limit of Riemann sums
3. Evaluate definite integrals using substitution
4. Find the area between two curves
5. Find and interpret Producer’s and Consumer’s Surplus’ from demand and supply equations
6. Be able to calculate and interpret the Gini Quotient from Lorentz Curves

Preparation of Lesson Assignments

Remember, this course covers an entire semester of work or the equivalent of a classroom course lasting 15 weeks. That means that each lesson in this course equals nearly a week of course work and will require the same time and effort on your part. Do not expect to complete each lesson in a single study session.

In order to receive the most rapid service, mail each lesson in one of the addressed envelopes as soon as the lesson is completed or use the electronic submission option (see Electronic Submission Options in the appendix for additional information).

General Instructions

A large part of the instructional process is conducted through the lesson assignments that are located at the end of each lesson. Follow the steps listed below when mailing assignments.

Type or write on one side of 8½ by 11” paper, leaving a one-inch margin on both sides for instructor notes.

Put your name, enrollment number, course number, and lesson number at the top right hand corner of each page. Number your pages 1 of __, etc.

Make a copy of your lessons in case any of them are lost in the mail.

Complete a lesson cover sheet (located in your course packet) for each lesson, and fold it so that your address is on the outside.

Submit one lesson per envelope. Failure to follow this procedure may result in your lesson not being recorded for grading and will require resubmission.

For each lesson, place the corresponding label on the envelope, and mail or bring to the IDL office.

Your lessons will be recorded according to the date received in the IDL office, not the date you mailed them.

IDL will only accept three lessons every seven calendar days.

Follow any additional instructions listed below.
Course Specific Instructions

Lesson reports may be written on ordinary notebook paper using a pencil or pen (not red ink). The grader will reject illegible work. Follow these steps in preparing your lessons:

1. Study the text and the comments and examples in this study guide before preparing your lesson report.

2. Write neatly on one side of the paper. Circle, box, or underline the solution. Show all supporting work.

3. Use standard 8½" × 11" paper.

4. Put your name, course name, enrollment number and lesson number in the upper right hand corner of each sheet.

5. Label each problem with the correct number and the textbook page from which it came.

6. Follow the directions in the “How to Take a College Independent Study Course” beginning on page iii of this guide, to prepare your lessons for mailing.

This is a self-taught course and as such requires that you possess some self discipline. Be conscientious. If a part of a lesson is marked incorrect, you should determine where you went wrong and correct your mistakes. Every effort must be made to work independently since your exams are taken without outside assistance. The instructor does not supply written answer keys to the lessons.

Suggested Study Techniques

Carefully study the textbook, study guide material (if applicable), additional resources provided, and the information in your course guide before you begin to prepare the lesson assignments. This study should include a detailed examination of the illustrative problems and examples, as well as the assigned reading. After a lesson assignment has been completed, a rapid re-reading of the related text and other materials is strongly recommended.

Review your lesson assignments after they have been graded and returned to you. LSU Independent & Distance Learning suggests that you wait for your first lesson to be returned to you before you submit subsequent lessons; however, after the first lesson, it is normally not necessary to wait for the corrected lesson assignment to be returned before completing and submitting the next one.

One temptation you may have in an independent study course is to rely too heavily on textbook material when preparing your lesson assignment. If you
give in to such a temptation, you may not realize until exam time that the perfect response you prepared was possible only because you repeatedly referred to the textbook without really learning or understanding the material. Therefore, you should attempt each assignment without referring to the textbook, and if “thumbing back” is necessary, be sure you have actually learned the point rather than merely reflected it in the answer.

Put yourself on a definite schedule. Set aside a certain block of hours per day or week for this course and work in a place where distractions are minimal. Try to submit a lesson each week or at least every two weeks. Delays in submitting lessons usually result in lagging interest and the inability to complete the course.

Academic Integrity

*LSU Independent & Distance Learning adheres to Louisiana State University’s policy on academic misconduct. This policy defines plagiarism as follows:*

“Plagiarism” is defined as the unacknowledged inclusion of someone else’s words, structure, ideas, or data. When a student submits work as his/her own that includes the words, structure, ideas, or data of others, the source of this information must be acknowledged through complete, accurate, and specific references, and, if verbatim statements are included, through quotation marks as well. Failure to identify any source (including interviews, surveys, etc.), published in any medium (including on the internet) or unpublished, from which words, structure, ideas, or data have been taken, constitutes plagiarism.¹

Contact Information

If you need to contact your instructor concerning your lesson assignment, you may include a note with your completed assignment, or you may email him or her at MyInstructor@outreach.lsu.edu. Your instructor does not have an office within the Independent & Distance Learning building. Instructors only answer questions related to course content. Please direct all other questions to our Learner Services office by emailing Iservices@outreach.lsu.edu or by calling 800-234-5046.

Examinations and Grading Policy

The course consists of 17 lessons and a mid-course and final examination. If the examinations are successfully completed, the exams will count as 68 percent of the final course grade, with lesson grades making up the remaining 32 percent. The exams are equally weighted at 34 percent each. The mid-course should be taken after Lesson 9. The focus of the final exam will be on the material covered from Lesson 10 to Lesson 17. Calculators are allowed on the exams, but students may not use programmable or graphing calculators. Three hours are allowed to complete the exams, which should be adequate time. The problems on the exams will be very similar to those in the lesson assignments.

There will be two examinations. The mid-course exam follows Lesson 9, and the final exam follows Lesson 17.

Lesson assignments count 100 points each. Exams are 100 points each.

Course grade = average of lesson assignments (.32) + exam scores (.68).

The following grading scale applies:

<table>
<thead>
<tr>
<th>Percentage Range</th>
<th>Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>90% - 100%</td>
<td>A</td>
</tr>
<tr>
<td>80% - 89%</td>
<td>B</td>
</tr>
<tr>
<td>70% - 79%</td>
<td>C</td>
</tr>
<tr>
<td>60% - 69%</td>
<td>D</td>
</tr>
<tr>
<td>0% - 59%</td>
<td>F</td>
</tr>
</tbody>
</table>

You must earn a passing average on the examinations in order to pass the course.

Transcript Information

After you have completed this course, your grade will be filed with the Office of the University Registrar. If a transcript is needed, it is your responsibility to make a request in writing to:

Office of the University Registrar
Louisiana State University
Thomas Boyd Hall
Baton Rouge, LA  70803
Phone: 225-578-1686
FAX: 225-578-5991
Examination Proctors

If you are not going to take your exam at LSU–Baton Rouge, notify us of your proctor by sending the completed Exam Proctor Information Form located in the appendix of this course guide to the Independent & Distance Learning office.

Please read the College Examination Information document in the appendix of this course guide for further details.
Lesson 1: Limits

Key Terms

- average velocity
- instantaneous velocity
- $x \to a$
- $x \to -\infty$
- average rate of change
- instantaneous rate of change
- limit of a function
- difference quotient
- $x \to a$
- $x \to \infty$

Lesson Introduction

If an object moves from a point $A$ to a point $B$ in a time period of length $t$, then we define the **average velocity** of the object to be the distance from $A$ to $B$ divided by $t$. To be more specific, suppose the position of the object is given by the position function $s(t)$ then the average velocity of the object in moving from $A$ to $B$, denoted by $\bar{v}$, is given by the quotient

$$\bar{v} = \frac{s(B) - s(A)}{t}.$$

Note that this agrees with the familiar formula $\text{rate} = \frac{\text{distance}}{\text{time}}$.

In a similar manner, we can define the average rate of change for an arbitrary function of $x$. Thus, the **average rate of change** of $f(x)$ as $x$ moves from $a$ to $a+h$ is given by

$$\text{Average Rate of Change} = \frac{\text{Change in } f}{\text{Change in } x} = \frac{f(a+h) - f(h)}{h}.$$  

The expression $\frac{f(a+h) - f(h)}{h}$ is called a **difference quotient**.
Lesson 1: Limits

Example 1

Let \( f(x) = x^2 + 2 \) and let \( x \) vary from \( x = 1 \) to \( x = 3 \). The average rate of change for \( f(x) \) will be

\[
\frac{f(3) - f(1)}{3-1} = \frac{11 - 3}{2} = \frac{8}{2} = 4.
\]

Example 2

Produce an expression for the average velocity of an object whose position function is given by \( s(t) = \sqrt{t} \) over an arbitrary interval from \( a \) to \( a + h \).

\[
\frac{s(a+h) - s(a)}{h} = \frac{\sqrt{a+h} - \sqrt{a}}{h} = \frac{\sqrt{a+h} - \sqrt{a}}{h} \cdot \frac{\sqrt{a+h} + \sqrt{a}}{\sqrt{a+h} + \sqrt{a}}
\]

This last step is actually rationalizing the numerator. The reason for doing this will be made clearer in the following lessons. We finish the example by simplifying the quotient.

\[
\frac{a+h-a}{h(\sqrt{a+h} + \sqrt{a})} = \frac{1}{\sqrt{a+h} + \sqrt{a}}
\]

If we consider taking the average velocity as elapsed time approaches zero, then we get a better idea about the velocity of an object at a particular instant. To compute the instantaneous velocity or the instantaneous rate of change for a function \( f(x) \) we form the difference quotient, reduce it, and then let \( h \) assume values successively closer to zero. The values of \( \frac{f(x+h) - f(x)}{h} \) will approach the instantaneous rate of change for \( f \).

Example 3

Let \( f(x) = x^2 + 2x + 2 \). The instantaneous rate of change for \( f(x) \) is computed by first simplifying the difference quotient.

\[
\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 + 2(x+h) + 2 - (x^2 + 2x + 2)}{h}
\]

\[
= \frac{x^2 + 2xh + h^2 + 2x + 2h + 2 - x^2 - 2x - 2}{h} = \frac{2xh + h^2 + 2h}{h} = 2x + 2 + h
\]
As $h$ is chosen progressively closer and closer to zero, we see that the simplified difference quotient will approach the expression $2x + 2$.

Consider the graph of the function $f(x) = x^2 + 1$, whose graph follows.

Since $y = f(x) = x^2 + 1$ is a function of $x$, it is sensible to ask how $y$ varies as $x$ is varied. In particular, if you choose $x$ very close to 0, what does this do to $y$? From the graph it is apparent that as $x$ gets closer to 0, the $y$ value on the graph, represented as the height of the point, will get arbitrarily close to 1. We now need a few definitions.

We say $x$ approaches $a$, written $x \to a$, if $x$ is chosen closer and closer to $a$, but never equal to $a$. We define the limit of a function $f$ as $x$ approaches $a$ to be $L$, written

$$\lim_{{x \to a}} f(x) = L,$$

provided that as $x$ gets arbitrarily close to $a$ (but not equal to $a$), $f(x)$ gets arbitrarily close to $L$. Note that $L$ must be a unique number.

**Example 4**

Let $f(x) = 2x$. The graph is shown below.
Lesson 1: Limits

It is apparent that as \( x \to 0 \), we have \( y \to 0 \). Thus we may write \( \lim_{x \to 0} 2x = 0 \).

Example 5

Consider the function whose graph is sketched below.

![Graph of a function](image)

We say \( \lim_{x \to 0} f(x) \) does not exist, since as you choose \( x \) near 0 from the positive side the \( y \) values remain near 1, but as \( x \) approaches 0 from the negative side, the \( y \) values remain near –1. Thus as \( x \to 0 \), the \( y \) values do not approach a unique value.

Study Theorem 1 on page 102 of your text. It will be shown that to evaluate a limit of a polynomial function you merely need to substitute the approached point into the function. This technique will work on rational functions when the expression does not become 0 in the denominator.

Example 6

Evaluate \( \lim_{x \to 2} x^2 + 2x + 4 \). This limit is evaluated using the comment above.

Since \( x^2 + 2x + 4 \) is a polynomial function, we can see that as \( x \) gets arbitrarily close to 2, the function gets arbitrarily close to \( 2^2 + 2(2) + 4 = 12 \). So we say the limit is 12.

Example 7

Evaluate \( \lim_{x \to 2} \frac{x-2}{x^2 - 4} \). If you try to evaluate the limit by evaluating the numerator and denominator separately, you get a 0 in the top and bottom. Proceed instead by simplifying algebraically.

\[
\lim_{x \to 2} \frac{x-2}{x^2 - 4} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}.
\]
Lesson 1: Limits

We define \( x \) **approaches infinity**, written \( x \to \infty \), to mean \( x \) assumes larger and larger positive values, increasing without bound. We say \( x \) **approaches negative infinity**, written \( x \to -\infty \), when we mean \( x \) decreases without bound. With this in mind, we define

\[
\lim_{x \to \infty} f(x) = L
\]

if \( f(x) \) approaches \( L \) as \( x \) increases without bound. A similar definition is applicable for the case of \( x \) approaching \( -\infty \). Towards this end, the student is referred to Theorem 2 on page 107.

**Theorem:** If \( p > 0 \) then \( \lim_{x \to \infty} \frac{1}{x^p} = 0 \) and \( \lim_{x \to -\infty} \frac{1}{x^p} = 0 \) provided \( x^p \) is defined for negative \( x \).

**Example 8**

Evaluate \( \lim_{x \to \infty} \frac{3x + 2}{6x - 5} \). Notice that the limit of the numerator and the limit of the denominator do not exist, since they each become arbitrarily large. That is, as \( x \to \infty \), both the numerator and the denominator increase without bound. Divide the numerator and the denominator by the highest power of \( x \) appearing in the denominator. Thus,

\[
\lim_{x \to \infty} \frac{3x + 2}{6x - 5} = \lim_{x \to \infty} \frac{3x}{6x} \frac{2}{x} = \lim_{x \to \infty} \frac{3 + \frac{2}{x}}{6 - \frac{5}{x}} = \frac{3}{6} = \frac{1}{2}
\]

Note we used the above theorem to conclude \( \lim_{x \to \infty} \frac{2}{x} = 0 \) and \( \lim_{x \to \infty} \frac{5}{x} = 0 \).

**Theorem:** \( f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0} \) is a rational function, then

\[
\lim_{x \to \pm \infty} f(x) = \begin{cases} 
0 & \text{if } n < m \\
\frac{a_n}{b_m} & \text{if } n = m \\
\text{Does not exist if } n > m
\end{cases}
\]
Lesson 1: Limits

Example 9

Evaluate \( \lim_{x \to \infty} \frac{3x^3 + 2x + 1}{5 - 2x^3} \) using the above theorem. Because the degree of the numerator equals the degree of the denominator, the limit is merely the quotient of the coefficients of the terms of highest degree. Thus,

\[
\lim_{x \to \infty} \frac{3x^3 + 2x + 1}{5 - 2x^3} = -\frac{3}{2}.
\]

Using the notation of limits, we state that the instantaneous rate of change for \( f \) at the point \( x \) is

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.
\]

Reading Assignment

Tan, section 2.4

Lesson Assignment

Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.

Exercise 2.4: 1–8, 10, 15, 17, 21, 23, 33, 41, 47, 49, 55, 57, 59, 65, 66, 73, 76, 78
Lesson 2: One-Sided Limits and Continuity: The Derivative

Key Terms

- $x \to a^+$, left-hand limit
- tangent line
- differentiable
- $x \to a^-$, right-hand limit
- secant line
- $f'(x)$, continuous derivative

Lesson Introduction

We say that \( x \) approaches \( a \), from the right, written as \( x \to a^+ \), if we let \( x \) get arbitrarily close to \( a \), but not equal to \( a \) and use only numbers to the right of \( a \). Thus, the \( x \) values are always larger than \( a \). In a similar fashion, we can define \( x \) approaches \( a \) from the left, written \( x \to a^- \), if \( x \) approaches \( a \), but not equal to \( a \) and we use only numbers to the left of \( a \). Thus, the \( x \) values are always less than \( a \). These notions can be used to define special types of limits called one-sided limits.

\( f(x) \) has a right-hand limit \( L \) as \( x \) approaches \( a \) from the right, written

\[
\lim_{x \to a^+} f(x)
\]

if the values of \( f(x) \) get arbitrarily close to \( L \) as \( x \) is chosen arbitrarily close to \( a \) from the right but with \( x \) not equal to \( a \).

\( f(x) \) has a left-hand limit \( L \) as \( x \) approaches \( a \) from the left, written

\[
\lim_{x \to a^-} f(x)
\]

if the values of \( f(x) \) get arbitrarily close to \( L \) as \( x \) is chosen arbitrarily close to \( a \) from the left but with \( x \) not equal to \( a \).

**Theorem:** Let \( f(x) \) be defined for all \( x \) near \( a \) with the possible exception of \( a \) itself. Then

\[
\lim_{x \to a^-} f(x) = L \text{ if and only if } \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L.
\]
Lesson 2: One-Sided Limits and Continuity: The Derivative

For the original limit to exist, both of the one-sided limits must exist and be equal.

Example 1

Consider the graph of the function below.

From the graph, we see that as $x$ approaches 0 from the left, the function values (the height of the graph) get arbitrarily close to 1. Thus, we write $\lim_{x \to 0^-} f(x) = 1$. From the right hand side, we see that the functional values get closer and closer to −1 as the $x$ values are allowed to approach 0 from the right. Thus, we see $\lim_{x \to 0^+} f(x) = -1$. Since the one-sided limits are different, we say the limit fails to exist at 0.

Example 2

Consider the function defined as $f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ -x & \text{if } x > 0 \end{cases}$

As $x$ approaches 0 from the right, we use the definition $f(x) = -x$, since the $x$ values are greater than 0. We see that the limit from the right must be 0. Approaching 0 from the left, we are using $x$ values that are less than 0. Thus, the definition of the function is $f(x) = x^2$. The functional values again approach 0. Since the limits exist from both the right and the left, we say the limit exists and is also equal to 0. The reader is advised to sketch the graph of this function for his or her own benefit.

Suppose that as $x$ approaches $a$, the values of $f(x)$ increase without bound. Then we say that the limit for the function is $\infty$. It is important for the student to realize that we do not mean the functional values are approaching some number called $\infty$. When we write $\lim_{x \to a} f(x) = \infty$, we are simply using this as a short way to express the notion that $f(x)$ is getting large without bound. The
symbol $\lim_{x \to a} f(x) = -\infty$ is used when the functional values of $f(x)$ decrease without bound.

**Example 3**

Consider the graph of function $f(x) = \frac{1}{x}$.

As $x$ approaches 0 from the right, the functional values of $f(x) = \frac{1}{x}$ increase without bound. In a similar manner, as $x$ tends toward 0 from the left side, the values of $f(x) = \frac{1}{x}$ tend to decrease without bound. Thus, we may write

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$
Lesson 2: One-Sided Limits and Continuity: The Derivative

Example 4

Contrast the previous example with the function \( f(x) = \frac{1}{x^2} \) having the following graph.

As \( x \) tends toward 0 from either the left or the right, we see that the values of the function tend to increase without bound. We may write \( \lim_{x \to 0} \frac{1}{x^2} = \infty \).

We are frequently interested in functions whose graphs have no jumps, breaks, or gaps. Practically speaking, a function is said to be \textit{continuous} if its graph can be drawn without lifting the pencil from the paper. This naive definition, although useful, is not very precise. A precise definition of continuity involves continuity of a function first at a specific point.

A function \( f(x) \) is continuous at a point \( a \) if and only if

1. \( f(a) \) is defined;
2. \( \lim_{x \to a} f(x) \) exists and is finite; and
3. \( \lim_{x \to a} f(x) = f(a) \).

The following examples demonstrate the importance of these three conditions.
Lesson 2: One-Sided Limits and Continuity: The Derivative

Example 5

Let \( f(x) = \frac{x}{x} \). This function is undefined at \( x = 0 \). Its graph follows.

As the reader can see, the gap at \( x = 0 \) corresponds to the point where \( f(x) \) is undefined. Whenever \( f(x) \) is undefined, there is a break in the graph.

Example 6

Let \( f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \). The graph of this function follows.

This function is clearly not continuous at \( x = 0 \). The first condition of continuity is satisfied since the function is defined at \( x = 0 \) to be \( 0 \). However, the limit as \( x \) tends to \( 0 \) is not finite. Thus, the second condition of continuity is violated. Example 2d on page 120 of your textbook shows another example of failure of a limit to exist.
Lesson 2: One-Sided Limits and Continuity: The Derivative

Example 7

As a last example of the continuity conditions, consider \( f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \) and its associated graph shown below.

![Graph](image)

This function fails to be continuous at \( x = 0 \), despite the facts that it is defined at \( x = 0 \) and the limit as \( x \) tends to 0 exists and is 1. The problem lies with the third condition. We have \( f(0) = 0 \), but \( \lim_{x \to 0} f(x) = 1 \). Since these do not equal, the third condition fails.

A function is said to be continuous on \((a,b)\) if it is continuous at every point of \((a,b)\).

Example 8

Consider the three functions \( f(x) = x^2 + 3x - 2 \), \( g(x) = \sqrt{2x - 5} \) and \( h(x) = \frac{1}{(x-1)(x-3)} \).

The first function, \( f(x) \), is a polynomial and hence continuous for all real numbers. The function \( g(x) \) is continuous where it is defined. Hence, \( g(x) \) is continuous for \( x \geq \frac{5}{2} \). The last function, \( h(x) \), is continuous for all \( x \) except 3 and 1. These values would involve a division by 0. Study the properties of continuous functions on page 121 of the textbook.
Consider the figure below.

You can see that the slope of the line connecting \( A \) and \( B \) is given by the difference quotient

\[
\frac{f(x+h) - f(x)}{h}
\]

Any such line joining two points on a graph is called a secant line. If we allow \( h \) to approach 0, then point \( B \) on the graph will move closer to point \( A \). The limiting position of the secant line is called the tangent line to \( f(x) \) at the point \( A \).

The tangent line has a slope given by the limiting value of the difference quotient as \( h \) tends to 0.
Example 9

Determine the equation of the tangent line for the function \( f(x) = \frac{1}{x} \) at the point \( x = 2 \). We begin by computing the slope of the tangent line using the formula

\[
\text{slope of tangent line} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.
\]

Calculating, we get

\[
\text{slope of tangent line} = \lim_{h \to 0} \frac{1}{x+h} - \frac{1}{x} = \lim_{h \to 0} \frac{x}{(x+h)x} - \frac{(x+h)}{h} = \frac{-1}{x^2}.
\]

If we evaluate this expression at \( x = 2 \), we see that the slope of the tangent line is \( -\frac{1}{4} \). So we are left with the task of writing the equation of a line through the point \((2, \frac{1}{2})\) with slope \( -\frac{1}{4} \). This line is, using the slope intercept form of the linear equation, \( y - \frac{1}{2} = -\frac{1}{4}(x - 2) \).

The slope of the tangent line and the instantaneous rate of change for a function discussed in Lesson 1 are the same. This important notion is given a special name. We define the **derivative** of \( f(x) \) with respect to \( x \) to be the limit of the difference quotient as \( h \) tends to 0. The derivative is symbolized as \( f'(x) \), read "\( f \) prime," and so we write

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

provided that this limit exists.
Lesson 2: One-Sided Limits and Continuity: The Derivative

**Example 10**

Compute the derivative for \( f(x) = x^2 + 3x - 4 \) at the point \( x = 1 \). Using the definition, we get

\[
\begin{align*}
    f'(x) &= \lim_{h \to 0} \frac{(x + h)^2 + 3(x + h) - 4 - (x^2 + 3x - 4)}{h} \\
    &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - 4 - x^2 - 3x + 4}{h} \\
    &= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h} \\
    &= \lim_{h \to 0} (2x + h + 3) = 2x + 3
\end{align*}
\]

Evaluating this general derivative at the point \( x = 1 \) gives us \( f'(x) = 5 \).

A function is said to be **differentiable** at \( x \) provided \( f'(x) \) exists. A function may fail to have a derivative at a point as the following examples demonstrate.

**Example 11**

Let \( f(x) = x^{\frac{2}{3}} \), whose graph is shown below.

Recall that the slope of a vertical line is undefined. This function has a vertical tangent line at \( x = 0 \); hence, the function is not differentiable there.
Lesson 2: One-Sided Limits and Continuity: The Derivative

Example 12

Let \( f(x) = |x| \), whose graph is sketched below.

![Graph of |x|](image)

The function fails to have a derivative at the point \( x = 0 \) because the graph has an abrupt change of direction at the origin. This will always occur when a corner is present in a graph.

**Theorem:** If a function is differentiable at a point \( a \), then it is continuous at \( a \).

This theorem tells us that a function will fail to be differentiable at any point of discontinuity. Note that continuity does not imply the existence of a derivative, as evidenced by the example above. The absolute value function is continuous at the origin but not differentiable there.

Reading Assignment

Tan, sections 2.5 and 2.6

Lesson Assignment

Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.

Exercise 2.5: 1, 2, 4, 7, 8, 15–20, 27, 37, 40, 43, 44, 50, 51, 57, 61, 65

Exercise 2.6: 1, 5, 9, 12, 16, 26, 32, 47, 49, 55
Lesson 3: Basic Rules of Differentiation: The Product and Quotient Rules

Key Terms

- differentiation
- product rule
- quotient rule

Lesson Introduction

Recall from the previous lesson that the derivative of \( f(x) \), written \( f'(x) \) is computed using the limit of a difference quotient as

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

provided the limit exists. The process of calculating a derivative is called differentiation. We now present a number of theorems which will greatly simplify the act of differentiation.

**Theorem:** If \( f(x) = k \), where \( k \) is a constant, then \( f'(x) = 0 \).

This should be obvious because the graph of a constant function is a horizontal line. The derivative is the slope of the tangent line, and the tangent line is clearly horizontal as well.
Lesson 3: Basic Rules of Differentiation: The Product and Quotient Rules

**Example 1**

If \( f(x) = \pi \), \( g(x) = 2 \) and \( h(x) = \sqrt{7} \) then \( f'(x) = g'(x) = h'(x) = 0 \)

**Theorem:** If \( f(x) = x^r \) where \( r \) is any real number then \( f''(x) = rx^{r-1} \).

**Example 2**

If \( f(x) = \sqrt{x} \), \( g(x) = x^2 \) and \( h(x) = \frac{1}{x^3} \), then

\[
\begin{align*}
  f(x) &= \sqrt{x} = x^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2} x^{-1/2} \\
  g(x) &= x^2 \quad \Rightarrow \quad g'(x) = 2x \\
  h(x) &= \frac{1}{x^3} = x^{-3} \quad \Rightarrow \quad h'(x) = -3x^{-4}
\end{align*}
\]

**Theorem:** If \( f(x) = g(x) + h(x) \), then \( f'(x) = g'(x) + h'(x) \) and if \( f(x) = g(x) - h(x) \) then \( f'(x) = g'(x) - h'(x) \).

**Theorem:** If \( g(x) = k \cdot f(x) \), then \( g'(x) = k \cdot f'(x) \)

These rules are normally stated as: “The derivative of a sum is the sum of the derivatives”; “The derivative of a difference is the difference of the derivatives”; and “The derivative of a constant times a function is the constant times the derivative of the function.”

**Example 3**

If \( f(x) = x^3 + 7x^2 + 9x + 8 \) and \( g(x) = \sqrt{x^5} + \frac{5}{x^5} \), then \( f'(x) = 3x^2 + 7(2x) + 9 \)

and \( g'(x) = \frac{3}{2} x^{3/2} + 5(-5)x^{-6} \).

**Example 4**

If \( f(x) = x^3 - x^2 + 4 \), then determine all \( x \) such that the tangent line is horizontal at \( x \). We seek all \( x \) such that \( f'(x) = 0 \). By the rules of differentiation we see \( f''(x) = 3x^2 - 2x = x(3x - 2) \). By examination we see that \( f'(0) = f'(2/3) = 0 \).
Lesson 3: Basic Rules of Differentiation: The Product and Quotient Rules

**Example 5**

What is the instantaneous rate of change for \( f(x) = 3\sqrt{x} + 7x^3 \) at \( x = 1 \)? By definition, the instantaneous rate of change for a function is the derivative of the function. Thus, \( f'(x) = \frac{3}{2}x^{-\frac{1}{2}} + 21x^2 \) and \( f''(1) = \frac{3}{2} + 21 = \frac{45}{2} \).

There are several alternate notations employed for the derivative of \( y = f(x) \). Some of these are detailed below.

\[
f''(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(f(x)) = D_x f(x)
\]

The derivatives of products and quotients are somewhat more difficult. The following theorem details these rules and demonstrates the use of the aforementioned notation.

**Theorem: The Product Rule and the Quotient Rule**

\[
D_x \left( f(x)g(x) \right) = f'(x)g(x) + f(x)g'(x) \quad \text{and} \quad D_x \left( \frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}
\]

**Example 6**

\[
D_x \left[ (2x + 3)(3x - 2) \right] = (2x + 3)'(3x - 2) + (2x + 3)(3x - 2)'
\]

\[
= (2)(3x - 2) + (2x + 3)(3) = 6x - 4 + 6x + 9 = 12x + 5.
\]

**Example 7**

\[
D_x \left[ \frac{x^3 + 2x + 7}{x^4 - 4x} \right] = \frac{(x^4 - 4x)(3x^2 + 2) - (x^3 + 2x + 7)(4x^3 - 4)}{(x^4 - 4x)^2}
\]
Lesson 3: Basic Rules of Differentiation: The Product and Quotient Rules

Example 8

If \( f(x) = \frac{x}{x^2 + 1} \), then determine the points where \( f(x) \) has a horizontal derivative.

\[
f'(x) = \frac{(x^2 + 1)(1) - (x)(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} = 0
\]

It is apparent that \( f'(x) = 0 \) when \( x = \pm 1 \).

Reading Assignment

Tan, sections 3.1 and 3.2

Lesson Assignment

Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.

Exercise 3.1:  1–35 odd, 41, 44, 48
Exercise 3.2:  1–30 odd, 32, 37, 43, 47, 49
Lesson 4: The Chain Rule

Key Terms
composition
chain rule
general power rule

Lesson Introduction

The function \( h(x) = \sqrt{x^2 + 1} \) can be viewed as a composition of the two functions \( f(x) = \sqrt{x} \) and \( g(x) = x^2 + 1 \). In particular,
\[
f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1} = h(x).
\]

Example 1

Express \( h(x) = 3\left(x^2 - 5\sqrt{x}\right)^7 \) as a composition of simpler functions. Let \( f(x) = 3x^7 \), and choose \( g(x) = x^2 - 5\sqrt{x} \). We now compose \( f(x) \) and \( g(x) \) to reveal
\[
f(g(x)) = f(x^2 - 5\sqrt{x}) = 3(x^2 - 5\sqrt{x})^7 = h(x).
\]

An expression of the form \( (h(x))^r \) is frequently encountered in calculus. This function is easily expressed as the composition of \( f(x) = x^r \) and \( h(x) \). The following important theorem will enable us to differentiate compositions of functions.

**Theorem: The Chain Rule:** If \( f(x) = g(h(x)) \) is the composition of two differentiable functions \( g \) and \( h \), then \( f \) is differentiable, and
\[
f'(x) = g'(h(x)) \cdot h'(x)
\]
Lesson 4: The Chain Rule

Thus, to find \( f'' = \left( g\left( h(x) \right) \right)' \), you differentiate \( g(x) \) then replace the \( x \) in \( g'(x) \) by \( h(x) \), finally multiplying by \( h'(x) \).

**Example 2**

Let \( f(x) = (x^3 + 10)^{53} \). This can be written as \( h(g(x)) \), with \( h(x) = x^{53} \) and \( g(x) = x^3 + 10 \). Now \( h'(x) = 53x^{52} \) and \( g'(x) = 3x^2 \). Thus, by the chain rule we arrive at

\[
f''(x) = h'(g(x))g'(x) = 53(x^3 + 10)^{52} \cdot 3x^2
\]

**Example 3**

Let \( h(x) = \sqrt[3]{x^2 + 2x + 1} \). Use the chain rule to compute \( h' \). Let \( f(u) = \sqrt[3]{u} \) and \( g(x) = x^2 + 2x + 1 \), then \( h = f(g(x)) \). So, by the chain rule we get

\[
h' = f'(g(x)) \cdot g'(x).
\]

Now \( f'(u) = \frac{1}{3}u^{-\frac{2}{3}} \), so \( f'(g(x)) = \frac{1}{3}(x^2 + 2x + 1)^{-\frac{2}{3}} \).

Combining, we arrive at \( h' = f'(g(x)) \cdot g'(x) = \frac{1}{3}(x^2 + 2x + 1)^{-\frac{2}{3}}(2x + 2) \).

**Theorem: The General Power Rule:**

If \( f(x) = (g(x))^n \), then \( f''(x) = n(g(x))^{n-1} g'(x) \).

**Example 4**

If \( f(x) = (3x^2 + 2)^{100} \), then \( f''(x) = 100(3x^2 + 2)^{99}(6x) \).

If \( f(x) = \sqrt{x^4 + 7} \), then \( f'(x) = \frac{1}{2}(x^4 + 7)^{-\frac{1}{2}}(4x^3) = \frac{2x^3}{\sqrt{x^4 + 7}} \).
Lesson 4: The Chain Rule

Example 5

\[
D_x \left( \frac{1}{\sqrt{5x^3 + 2}} \right) = D_x \left( (5x^3 + 2)^{-\frac{1}{2}} \right) = \frac{-1}{2} \left( 5x^3 + 2 \right)^{-\frac{3}{2}} (15x^2) = \frac{-15x^2}{2 \left( 5x^3 + 2 \right)^{\frac{3}{2}}}
\]

Using \( \frac{dy}{dx} \) notation for the derivative allows us to write the chain rule in a different form. Let \( y = g(u) \) and \( u = h(x) \) be differentiable functions, then the chain rule can be expressed as

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.
\]

Example 6

If \( y = 17u^3 + 3u^2 \) and \( u = \sqrt{x} = x^{\frac{1}{2}} \), then we get

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (51u^2 + 6u) \cdot \frac{1}{2} x^{-\frac{1}{2}} = \left( 51x + 6\sqrt{x} \right) \cdot \frac{1}{2\sqrt{x}} = \frac{51}{2} \sqrt{x} + 3.
\]

Reading Assignment

Tan, section 3.3

Lesson Assignment

Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.

Exercise 3.3: 1, 4, 9, 15, 18, 21, 24, 27, 33, 40, 43, 49, 53, 59
Lesson 5: Marginal Functions in Economies: Higher-Order Derivatives

Key Terms

- cost
- marginal cost
- average cost
- economies of scale
- elastic
- second derivative
- revenue
- marginal revenue
- fixed costs
- demand function
- inelastic
- $n^{th}$ derivative
- profit
- marginal profit
- variable costs
- elasticity of demand
- unitary

Lesson Introduction

Economists use the following functions in their study of business.

\[ C(x) = \text{cost of producing } x \text{ items} \]
\[ R(x) = \text{revenue from the sale of } x \text{ items} \]
\[ P(x) = \text{profit from the production and sale of } x \text{ items} \]

The equation relating these functions is \( P(x) = R(x) - C(x) \). In words, profit is revenue minus cost.
Lesson 5: Marginal Functions in Economies: Higher-Order Derivatives

An economist uses the word *marginal* to refer to the rate of change. That is, the word marginal means derivative. Thus,

\[ C'(x) = \text{marginal cost} \]
\[ R'(x) = \text{marginal revenue} \]
\[ P'(x) = \text{marginal profit} \]

**Example 1**

A company sells pans for $3 each, and the costs associated with these pans are given by \( C(x) = 1.25x + .01x^2 + 50 \). The marginal cost function is \( C'(x) = 1.25 + .02x \). The revenue function is \( R(x) = 3x \), hence marginal revenue is \( R'(x) = 3 \). Marginal profit is

\[ P'(x) = R'(x) - C'(x) = 3 - (1.25 + .02x) = 1.75 - .02x. \]

The value of the marginal cost at \( x = x_0 \) is an approximation to the additional cost incurred in producing one unit beyond the \( x_0 \) level of production. This is true because \( C'(x_0) \) is really the slope of the tangent line to the cost curve \( C(x) \) at the point \( x = x_0 \). Thus, the derivative can approximate the change in the height of the function.

**Example 2**

Assume \( C(x) = .00002x^3 - .48x^2 + 300x + 150,000 \). Compute \( \Delta C \) (the change in \( C \)) as \( x \) moves from 100 to 101 and then compute \( C'(100) \).

\[ \Delta C = C(101) - C(100) = 179,830.96 - 179,540 = 290.96. \]

Now differentiate to obtain \( C'(x) = .00006x^2 - .096x + 300 \) and evaluate at 100 to get \( C'(100) = 291 \). Notice how the value of the derivative at 100 is quite close to the actual change in the cost function.

The **average cost** per unit is defined as cost divided by the number of units produced. This quantity is symbolized as \( \overline{C}(x) \) and is given by the formula

\[ \overline{C}(x) = \frac{C(x)}{x}. \]
Example 3

The cost function for a company is \( C(x) = 500 + 35x \). The 500 represents what are known as fixed costs. **Fixed costs** are those costs that are independent of production level, such as rent, executive salaries, and insurance fees. The 35 represents variable costs. **Variable Costs** are those costs that are dependent on the level of production. The average cost function will be

\[
\overline{C}(x) = \frac{500}{x} + 35.
\]

The marginal average cost will be \( \overline{C}'(x) = -\frac{500}{x^2} \). Since the derivative of the average cost is negative for all \( x \), we see that the average cost is decreasing for all \( x \). This implies that the per unit cost is decreasing as the production quantity rises. This is referred to in economic circles as **economies of scale**.

Suppose the demand for a product is given as a function of price by the expression \( x = f(p) \). This **demand function** gives the quantity demanded of the product for a price \( p \). We define the **elasticity of demand** for a good with demand equation \( x = f(p) \), as the function \( E(p) = \frac{-pf''(p)}{f'(p)} \). Economists use the following terms to describe demand.

- The demand is **elastic** if \( E(p) > 1 \).
- The demand is **inelastic** if \( E(p) < 1 \).
- The demand is **unitary** if \( E(p) = 1 \).

The notion of elasticity of demand governs how responsive the demand for a product is in regard to changes in the price. The textbook details all of the pertinent cases in the box on page 203.

Example 4

Let the demand equation for a product be given by \( x = f(p) = \sqrt{400 - 5p} \). The elasticity of demand is calculated as follows.

\[
E(p) = \frac{-pf''(p)}{f'(p)} = \frac{-p \left( \frac{1}{2} (400 - 5p)^{-\frac{1}{2}} \right) (-5)}{\frac{1}{2} (400 - 5p)^{-\frac{1}{2}}} = \frac{5p}{2(400 - 5p)}.
\]
Lesson 5: Marginal Functions in Economies: Higher-Order Derivatives

If the price of the product is set at 40, then the elasticity of demand is \( \frac{1}{2} < 1 \).

This classifies the demand as inelastic. If the price increases past 40, we can expect the revenue to increase. This means that the price increases in the product are not enough to depress sales. Thus, increasing the price of the product will generate additional revenue. In contrast, a price of 60 will result in an elasticity of demand of \( 2 > 1 \). This classifies the demand as elastic. Price increases past 60 will result in a decrease in revenue. That is, price increases will drive off customers in sufficient quantity to depress revenue.

If \( f(x) \) is differentiated to obtain \( f'(x) \), it may be possible to compute a derivative of \( f'(x) \). If \( f''(x) \) is differentiated, the resulting function is called a second derivative and can be written \( f''(x) \). You can take succeeding derivatives as long as the resulting functions are differentiable. In general, we represent the \( n^{th} \) derivative of \( f(x) \) in one of the following ways.

\[
\text{\( n^{th} \) derivative } = f^{(n)}(x) = y^{(n)} = \frac{d^n y}{dx^n}.
\]

**Example 5**

Compute the first, second, and third derivatives of the function \( y = 7x^3 + 3x^2 + 18 \). Using our rules of differentiation, we get

\[
y'(x) = 21x^2 + 6x
\]
\[
y''(x) = 42x + 6
\]
\[
y'''(x) = 42.
\]

**Example 6**

Determine the \( n^{th} \) derivative for \( f(x) = \frac{1}{x} \). We begin by taking a few of the derivatives for the function to see if we can establish a pattern in the derivatives.

\[
y(x) = x^{-1}
\]
\[
y'(x) = -1x^{-2}
\]
\[
y''(x) = 2x^{-3}
\]
\[
y'''(x) = -6x^{-4}
\]
\[
y^{(4)}(x) = 24x^{-5}
\]
It seems that a pattern is apparent. Notice the derivatives alternate in sign. This can be achieved by including a term \((-1)^n\) in the \(n\text{th}\) derivative. Thus we may write

\[
f^{(n)}(x) = (-1)^n n(n-1)(n-2)\ldots(3)(2)(1)x^{-(n+1)}
\]

\[
= (-1)^n (n!)x^{-(n+1)},
\]

where we use the symbol \(n!\), read “\(n\) factorial,” to represent the product of the integers from \(n\) down to 1.

If \(s(t)\) denotes the position of a particle at time \(t\), we know that \(s'(t)\) is the instantaneous velocity of the particle. The second derivative of the position function, \(s''(t)\) is called the **acceleration** of the particle. Thus, the acceleration represents the rate of change for velocity with respect to time.

**Example 7**

An object is thrown into the air in such a way that its height in feet after \(t\) seconds is given by \(h(t) = -16t^2 + 82t + 14\). Determine the velocity and acceleration when time \(t\) is zero. By differentiating we get velocity \(s'(t) = -32t + 82\) and acceleration \(s''(t) = -32\). When \(t = 0\), we have \(s'(0) = 82\) and \(s''(0) = -32\). These values are called the initial velocity and the initial acceleration.

**Reading Assignment**

Tan, sections 3.4 and 3.5

**Lesson Assignment**

*Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.*

Exercise 3.4: 1, 4, 9, 11, 14, 23, 30
Exercise 3.5: 1, 4, 9, 15, 22, 29, 35, 36
Lesson 6: Applications of the First Derivative

Key Terms

increasing
local minimum
decreasing
local maximum
critical value

Lesson Introduction

We say $f(x)$ is increasing if, as $x$ increases, we have $f(x)$ increasing. Graphically, this means that as you move from left to right on the $x$-axis, the curve increases its height. Similarly, a function is decreasing if, as $x$ increases, the values of $f(x)$ decrease.

Example 1

In the function sketched above, we see the graph is increasing on the intervals $[a, b]$, $[c, d]$ and $[e, f]$. The function and its graph decrease on the intervals $[b, c]$ and $[d, e]$. 
Lesson 6: Applications of the First Derivative

**Theorem:** Let $f(x)$ be continuous and differentiable; then...

1) if $f'(x) > 0$ for all $x$ in an interval $(a,b)$, then $f(x)$ is increasing on $(a,b)$.

2) if $f'(x) < 0$ for all $x$ in an interval $(a,b)$, then $f(x)$ is decreasing on $(a,b)$.

**Example 2**

Let $f(x) = x^3 + x^2 - 5x - 5$. Differentiating, we get

$$f'(x) = 3x^2 + 2x - 5 = (3x + 5)(x - 1).$$

We are faced with the problem of determining where a function is positive and where it is negative. The author uses a slightly different method than what I am about to describe. We will construct a chart that will tell us how $f'(x)$ changes sign by considering how each factor of $f'(x)$ varies in sign.

<table>
<thead>
<tr>
<th>Sign of $f'(x)$</th>
<th>+</th>
<th>-</th>
<th>+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of $(x-1)$</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Sign of $(3x+5)$</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

The $+$ and $-$ signs indicate the algebraic sign of each factor in the bottom of the chart. To the left of $\frac{-5}{3}$, both factors are negative; hence, their product is positive. This is shown by the sign of $f'(x)$ at the top of the chart. Between $\frac{-5}{3}$ and 1, only one of the factors is negative, resulting in the derivative of the function being negative there. Lastly, both factors are positive to the right of 1. This gives us a positive derivative to the right of 1. Thus, the function is
increasing on the intervals \((-\infty,-\frac{5}{3}) \cup (1,\infty)\) and decreasing on the interval \([-\frac{5}{3},1]\). Notice that

\[ f(x) = x^3 + x^2 - 5x - 5 = (x^2 - 5)(x + 1) = \left( x - \sqrt{5} \right) \left( x + \sqrt{5} \right) (x + 1). \]

So our function has \(x\)-intercepts at \(x = -1, \sqrt{5}\) and \(-\sqrt{5}\). The \(y\)-intercept, obtained by letting \(x = 0\) is \(y = -5\). We plot these points and the points where \(f(x)\) changes from increasing to decreasing and vice-versa. Notice the curve changes direction at the values of \(x\) where the derivative changes sign. The graph of the function is shown here.

If \(f(x) \leq f(c)\) for all \(x\) near \(c\), we say that \((c, f(c))\) is a **local maximum** on the graph of \(y = f(x)\). Similarly, if \(f(x) \geq f(c)\) for all \(x\) near \(c\), we say that \((c, f(c))\) is a **local minimum** on the graph of \(y = f(x)\).

**Example 3**

Consider the function whose graph is shown below.
Lesson 6: Applications of the First Derivative

The function \( f(x) \) has local maxima at \((b, f(b))\) and \((d, f(d))\). The function has local minima at \((a, f(a)), (c, f(c))\) and \((e, f(e))\). Basically, a maximum is a mountain top and a minimum is a valley floor for the function.

We define \( x = c \), where \( c \) is a point in the domain of \( f(x) \), to be a critical value of \( y = f(x) \) if \( f'(x) = 0 \) or if \( f'(x) \) fails to exist. Critical values are very important in the remainder of our studies.

**Theorem:** If \( f(x) \) is continuous on the interval \((a, b)\), then any local maximum or minimum must occur at a critical value of \( f(x) \).

It is important to realize that this theorem does not say the function must have a local maximum or minimum at a critical value. It says “If there are any local extrema (a term meaning either local maxima or minima), they must occur at critical values.” The existence of critical values does not guarantee a local extrema.

**The First Derivative Test:** Let \( c \) be a critical value of \( f(x) \). If \( f''(x) < 0 \) for \( x < c \) and \( f''(x) > 0 \) for \( x > c \), then \((c, f(c))\) is a local minimum. If \( f''(x) > 0 \) for \( x < c \) and \( f''(x) < 0 \) for \( x > c \), then \((c, f(c))\) is a local maximum.

**Example 4**

Let \( f(x) = x^4 - 2x^2 + 2 \). Determine the intervals on which the function is increasing and decreasing and determine any local extrema. Differentiating, we get

\[
f''(x) = 4x^3 - 4x = (4x)(x - 1)(x + 1).
\]

The function clearly has critical values at \( x = 0, 1 \) and \(-1\). Determining the functional values associated with these critical values gives us \( f(0) = 2 \), \( f(1) = 1 \) and \( f(-1) = 1 \). We now construct a sign chart to help us classify the algebraic signs of the derivative.
Lesson 6: Applications of the First Derivative

From the sign chart, it is apparent this function has a local maximum at $x = 0$ and local minima at $x = 1$ and $x = -1$. The function is decreasing on the intervals $(-\infty, -1]$ and $[0, 1]$. The function is increasing on the intervals $[-1, 0]$ and $[1, \infty)$. A graph of this function is shown below.

**Reading Assignment**

Tan, section 4.1

**Lesson Assignment**

*Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.*

Exercise 4.1: 1, 3, 6, 8, 10, 13, 16, 21, 31, 35, 36, 40, 43, 47, 48, 57, 59, 73
Lesson 7: Applications of the Second Derivative

Key Terms

| concave down | second derivative test |
| possible inflection point | inflection point |
| concave up |

Lesson Introduction

Consider the two curves and their tangent lines below.

Curve I has the property that all tangent lines lie above the curve itself. Such a curve is said to be concave down. Another characterization is obtained by noting that when a curve is concave down and you move from left to right across the curve, the derivative decreases. Thus, if the derivative is decreasing on an interval, the curve is concave down on that interval. Curve II is said to be concave up. Curves that lie completely above their tangent lines are concave up. We note that if the derivative is increasing on an interval, the curve is concave up on that interval. A point on the graph where concavity changes is called an inflection point.
Lesson 7: Applications of the Second Derivative

Example 1

Examine the following graph.

The graph is concave down on the intervals \([a, b]\) and \([c, \infty)\). The graph is concave up on the intervals \((-\infty, a]\) and \([b, c]\). The graph has inflection points at \(a\), \(b\), and \(c\).

The following result gives a characterization of concavity in terms of the second derivative.

**Theorem:** Suppose \(f(x)\) is differentiable and \(f''(x)\) exists. If \(f''(x) > 0\) on the interval \((a, b)\) then \(f(x)\) is concave up on \((a, b)\). If \(f''(x) < 0\) on the interval \((a, b)\) then \(f(x)\) is concave down on \((a, b)\).

We need to remember the following information when considering concavity.

\[
\begin{align*}
  f''(x) > 0 & \quad \Leftrightarrow \quad \text{concave up} \\
  f''(x) < 0 & \quad \Leftrightarrow \quad \text{concave down}
\end{align*}
\]

Example 2

Let \(f(x) = x^3 + x^2 - 5x - 5\). Taking derivatives, we get

\[
\begin{align*}
  f'(x) = 3x^2 + 2x - 5 \quad \text{and} \quad f''(x) = 6x + 2 = 2(3x + 1).
\end{align*}
\]

A sign chart for the second derivative is shown with CD representing concave down and CU meaning concave up.
Lesson 7: Applications of the Second Derivative

\[ f''(x) = 2(3x + 1) \]

From the sign chart we can see the graph is concave up on the interval \((-\frac{1}{3}, \infty)\) and concave down on the interval \((-\infty, -\frac{1}{3})\).

Recall that a point on the graph where concavity changes is called an inflection point. Inflection points can occur only at those points where the second derivative is zero or fails to exist. The points where the second derivative is zero or fails to exist are not guaranteed to be inflection points. Observe the second derivative of \( f(x) = x^4 \) is zero at the origin, but there is no inflection point at the origin. These points where the second derivative is zero or fails to exist are called possible inflection points, because they are the only possibilities for points of inflection.

Example 3

Locate any inflection points on \( f(x) = \frac{x^4}{12} - 2x^2 \). Differentiating gives us

\[ f''(x) = x^2 - 4 = (x - 2)(x + 2). \]

Thus, the possible inflection points are \( x = 2 \) and \( x = -2 \). A sign chart for the second derivative will enable us to classify the intervals over which the function is concave up and those intervals over which the function is concave down. The sign chart follows.
Lesson 7: Applications of the Second Derivative

The sign chart makes it clear that the possible inflection points, \( x = 2 \) and \( x = -2 \), are indeed true inflection points.

**Example 4**

Determine the inflection points and the intervals where the function 

\[ f(x) = \frac{1}{x^2 + 1} \]

is concave up or concave down. We begin by finding the second derivative and factoring it completely.

\[
\begin{align*}
  f'(x) &= -1(x^2+1)^{-2}(2x) = \frac{-2x}{(x^2+1)^2}.
  \\
  f''(x) &= \frac{(x^2+1)^2(-2) - (-2x)(2(x^2+1)(2x))}{(x^2+1)^4} = \frac{(x^2+1)(-2x^2+8x^2)}{(x^2+1)^4} = \frac{6x^2}{(x^2+1)^3}.
\end{align*}
\]

The reader is left to complete the sign chart for the second derivative. It can be seen that the function has inflection points when \( x = \pm \frac{1}{\sqrt{3}} \). The function is concave up on the intervals \( (-\infty, -\frac{1}{\sqrt{3}}) \) and \( (\frac{1}{\sqrt{3}}, \infty) \). The function is concave down on the interval \( (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \).

A graph of the function is shown.

---

**Theorem: The Second Derivative Test:** Suppose \( f(x) \) is differentiable and 

\[ f''(c) = 0. \]

If \( f''(c) = 0 \), then \( f(x) \) has a local minimum at \( (c, f(c)) \). If 

\[ f''(c) < 0, \]

then \( f(x) \) has a local maximum at \( (c, f(c)) \).
Lesson 7: Applications of the Second Derivative

Thus, the algebraic sign of the second derivative at a point where \( f'(c) = 0 \) is a useful way to classify extrema without resorting to the first derivative test.

**Reading Assignment**

Tan, section 4.2

**Lesson Assignment**

*Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.*

Exercise 4.2: 1, 4, 7, 8, 10a, 12, 16, 24, 27, 31, 34, 44, 60, 69, 72
Lesson 8: Curve Sketching

**Key Terms**

vertical asymptotes  
horizontal asymptotes

**Lesson Introduction**

Examine the graph of \( f(x) = \frac{1}{x^2 - 1} \).

Notice that as \( x \) is chosen closer and closer to 1 or to \(-1\), the graph of the function tends to infinity or negative infinity. When this occurs, we say the function has a **vertical asymptote**. To be precise, we write that the vertical line \( x = a \) is a vertical asymptote for \( f(x) \) if

\[
\lim_{x \to a^+} f(x) = \pm \infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm \infty
\]

If the function in question is rational, we can easily determine its vertical asymptotes by finding the roots of the denominator. If \( f(x) = \frac{p(x)}{q(x)} \) is a quotient of two polynomials and if \( x = c \) is a root of \( q(x) \), then \( x = c \) is a vertical asymptote for \( f(x) \), provided \( p(c) \) is not zero at \( c \) as well.
Lesson 8: Curve Sketching

Example 1

The function \( f(x) = \frac{1}{x-4} \) has a vertical asymptote at \( x = 4 \), since the functional values get arbitrarily large near the root of the denominator. The function \( g(x) = \frac{3x}{(x-2)(x+3)} \) has vertical asymptotes at \( x = 2 \) and at \( x = -3 \). These are the roots of the denominator, and they are not simultaneously roots of the numerator. The function \( h(x) = \frac{3x}{x^2 + 1} \) has no vertical asymptotes since it is defined for all \( x \). Lastly, the function \( m(x) = \frac{x-2}{x^2 - 4} \) has a vertical asymptote at \( x = -2 \), but not at \( x = 2 \). There is no vertical asymptote at \( x = 2 \) because \( 2 \) is a simultaneous root of the numerator and the denominator.

Consider the graph of the following function.

Notice that as \( x \) moves to the right, the graph appears to flatten out, becoming more and more like the line \( y = 1 \). Conversely, as it moves to the left, the graph seems to become more like \( y = -1 \). These lines become what are known as horizontal asymptotes.

The line \( y = b \) is a horizontal asymptote for \( f(x) \) if

\[
\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to \infty} f(x) = b
\]

Horizontal asymptotes are merely the limits of functions at infinity or minus infinity. The calculation of these limits for rational functions was covered in Lesson 2.
Example 3

The function \( f(x) = \frac{x}{x^2 - 1} \) has a horizontal asymptote at \( y = 0 \). The degree of the numerator exceeds the degree of the denominator, so as \( x \) increases, we know the functional values decrease toward zero. The function 
\[
g(x) = \frac{2x^2 - 3x + 5}{9 + 5x - 7x^2}
\]
has a horizontal asymptote at \( y = -\frac{2}{7} \).

We see the limit at infinity for \( g(x) \) is \( -\frac{2}{7} \) because when the degree of the numerator and the denominator are equal, the limit is the quotient of the lead coefficients.

A polynomial, being a continuous function, has no vertical or horizontal asymptotes.

Study the graphing strategy on page 288 of the textbook. The following example will employ these techniques to sketch the graph of a rational function.

Example 4

Let \( f(x) = \frac{x}{x^2 - 1} \). The domain of the function is the set of all real numbers except for \( x = 1 \) and \( x = -1 \). The function is undefined at those values. Substituting \( x = 0 \) to find the \( y \)-intercept, reveals a value of 0 for the \( y \)-intercept. Since this fraction assumes a value of zero when \( x = 0 \), we see the only \( x \)-intercept for the function is 0. Since the degree of the numerator is smaller than the degree of the denominator, we know the horizontal asymptote for \( f(x) \) is \( y = 0 \). Vertical asymptotes occur at the zeros of the denominator, namely \( x = 1 \) and \( x = -1 \). Differentiating, we get

\[
f'(x) = \frac{(x^2 - 1)(1) - (x)(2x)}{(x^2 - 1)^2} = \frac{-1 - x^2}{(x^2 - 1)^2}.
\]

Notice this derivative is negative for all \( x \). This means the function has no critical values and no possible extrema, and the function is decreasing for all \( x \) for which the function is defined. In particular, the function is decreasing on the intervals \(( -\infty, -1), (1,1) \) and \((1,\infty)\).
Computing the second derivative gives us

\[ f''(x) = \frac{(x^2 - 1)^2(-2x) - (-1 - x^2)(2(x^2 - 1))(2x)}{(x^2 - 1)^4} \]

\[ = \frac{(x^2 - 1)(-2x) - (-1 - x^2)(2)(2x)}{(x^2 - 1)^3} = \frac{-2x^3 + 2x + 4x + 4x^3}{(x^2 - 1)^3} \]

\[ = \frac{6x + 2x^3}{(x^2 - 1)^3} = \frac{(2x)(3 + x^2)}{(x-1)^3(x+1)^3} \]

We see that the possible inflection points for \( f(x) \) are \( x = 1, x = -1 \) and \( x = 0 \). A sign chart for the second derivative follows.

\[
\begin{array}{c|cccccccc}
         & - & + & - & + \\
(x + 1) & - & - & + & + & + & + & + & + \\
(x - 1)  & - & - & - & - & - & - & - & + & + & + \\
2x       & - & - & - & - & - & + & + & + & + & + \\
-1 & 0 & 1 \\
\end{array}
\]

We see that the function is concave up on the intervals \((-1, 0)\) and \((1, \infty)\) and concave down on the intervals \((-\infty, -1)\) and \((0, 1)\). Because concavity changes at \( x = -1, x = 1 \) and \( x = 0 \), we classify these points as inflection points. The graph for the function is sketched below.
Lesson 8: Curve Sketching

Reading Assignment
Tan, section 4.3

Lesson Assignment

Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under "Preparation of Lesson Assignments" in the course syllabus. This lesson assignment is worth 100 possible points.

Exercise 4.3: 1, 2, 7, 10, 15, 20, 25, 29, 38, 43, 49, 54, 58, 61
Lesson 9: Optimization

Key Terms

- absolute maximum
- absolute minimum
- auxiliary relationship

Lesson Introduction

\( f(c) \) is called an **absolute maximum** of \( y = f(x) \) if \( f(c) \geq f(x) \) for all \( x \).

\( f(c) \) is called an **absolute minimum** of \( y = f(x) \) if \( f(c) \leq f(x) \) for all \( x \).

The student is advised to study the figure on page 299.

**Theorem:** If \( f(x) \) is continuous on a closed interval \([a, b]\), then \( f(x) \) attains an absolute maximum and an absolute minimum on \([a, b]\). These absolute extrema of \( f(x) \) will occur at a critical point or at an end point.

To find absolute extrema, we find all critical points, then evaluate \( f(x) \) at the endpoints of the interval and at the critical values. The largest value of \( f(x) \) will be the absolute maximum, and the smallest will be the absolute minimum.

**Example 1**

Let \( f(x) = x^4 - 5x^2 + 4 \) where \( x \) must lie in the closed interval \([0, 2]\). We first compute \( f''(x) = 4x^3 - 10x = 4x \left( x^2 - \frac{5}{2} \right) = 4x \left( x - \frac{\sqrt{5}}{2} \right) \left( x + \frac{\sqrt{5}}{2} \right) \). Thus, the critical numbers of \( f(x) \) are \( x = 0 \) and \( x = \pm \frac{\sqrt{5}}{2} \). Notice \( -\frac{\sqrt{5}}{2} \) is not in the interval of consideration, so

\[
 f(0) = 4, \quad f \left( \frac{\sqrt{5}}{2} \right) = -\frac{9}{4} \quad \text{and} \quad f(2) = 0.
\]
Lesson 9: Optimization

This function has an absolute maximum at \((0, 4)\) and an absolute minimum at \(\left(\frac{5}{2}, -\frac{9}{4}\right)\) on the interval \([0, 2]\).

**Theorem:** If \(f(x)\) is continuous on an interval I and \(c\) is the only critical value of \(f(x)\) in I, then if \(f''(c) > 0\), \(f(c)\) is an absolute minimum and if \(f''(c) < 0\), then \(f(c)\) is an absolute maximum.

**Example 2**

Let \(f(x) = x^5 - 5x^3\) on the interval \((0, \infty)\). We know

\[
f''(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3) = 5x^2(x - \sqrt{3})(x + \sqrt{3}).
\]

Thus the critical values of \(f(x)\) are \(x = 0\) and \(x = \pm \sqrt{3}\), but only \(\sqrt{3}\) lies in the interval \((0, \infty)\). Now \(f'(x) = 20x^3 - 30x\) and \(f''(\sqrt{3}) = 20(\sqrt{3})^3 - 30\sqrt{3} \approx 52 > 0\). Thus, by the above theorem, we know that \((\sqrt{3}, f(\sqrt{3})) \approx (\sqrt{3}, -10.392)\) is an absolute minimum for \(f(x)\) on the interval \((0, \infty)\).

We will now work a few applied problems employing maxima and minima analysis. The student is advised to study the text’s guidelines for solving applied problems on pages 312 and 313.

**Example 3**

A veterinarian has 100 feet of fencing and wishes to construct six dog kennels by first building a fence around a rectangle region, and then sub-dividing that region into six smaller rectangles by placing five fences parallel to one side. What dimensions will maximize the area? Draw the kennel as shown.
Label two of the sides $x$ and $y$. We seek to maximize area, so we need a formula for the area. This is, obviously, $A = xy$. Because $A$ is a function of two independent variables, we need a relationship between $x$ and $y$ that will allow us to replace one variable with the other. Such a relationship is called an **auxiliary relationship**. Because there is 100 feet of fence and we have two sides of length $x$ and seven sides of length $x$, we know $2y + 7x = 100$ or $y = \frac{100 - 7x}{2}$. We substitute this into the area formula to obtain $A = xy = x\left(\frac{100 - 7x}{x}\right)$. We can differentiate $A = \frac{1}{2}(100x - 7x^2)$ to get $A' = \frac{1}{2}(100 - 14x) = \frac{1}{28}\left(\frac{100}{14} - x\right)$. Thus, $x = \frac{100}{14} = \frac{50}{7}$ is the only critical number of $A$. $A''(x) = \frac{1}{2}(-14) < 0$. Thus, $x = \frac{50}{7}$ is the point at which area is a maximum.

$Y = \frac{1}{2}(100 - 7x) = \frac{1}{2}\left(100 - 7\left(\frac{50}{7}\right)\right) = 25$. Thus, the dimensions of the kennel of maximum area are 25 feet by $\frac{50}{7}$ feet.

**Example 4**

A company sells shoes to dealers at $20 per pair if fewer than 50 pairs are ordered. If 50 or more pairs are ordered (up to 600), the price per pair is reduced 2 cents times the number ordered. What size order produces maximum revenue for the company?

Revenue = (price)(number sold), and we let $x$ be order size.

Thus, Revenue = $R(x) = 20x$ if $x \in [0, 50]$, and, taking the quantity discount into consideration, we get $R(x) = (20 - .02x)(x)$ if $x \in [51, 600]$. For $x \in [0, 50]$, we have $R(x) = 20x$, and it is obvious that revenue is a maximum when $x = 50$, producing a revenue of $(20)(50) = 1000$.

For $x \in [51, 600]$ we must use calculus to locate the maximum for the revenue function.

Since $R(x) = (20 - .02x)(x) = 20x - (.02x^2)$, we have $R'(x) = 20 - .04x$ and $R''(x) = -.04$. 


Lesson 9: Optimization

The critical number for revenue is \( \frac{20}{0.04} = 500 \), and since \( R''(x) < 0 \) for all \( x \), we know that \( R(x) \) has a maximum at \( x = 500 \).

\[ R(500) = (20 - 0.02(500))(500) = 5000. \] \( R(500) > R(50) \), so we know that the order size producing the most income for the company is a 500-pair order.

Example 5

A printed page must contain 72 square inches of printing with margins of 2 inches on each side and 1 inch at top and bottom. What dimensions will minimize the paper used? See the drawing below.

We wish to find the minimum value of area \( A = (x + 4)(y + 2) \), subject to the condition the printed area \( P = 72 = xy \). We can solve the last equation for \( y \) to reveal \( y = \frac{72}{x} \). This can be substituted into the area formula to get

\[ A = (x + 4)
\left(
\frac{72}{x} + 2
\right)
= 72 + 2x + \frac{288}{x} + 8. \]

We differentiate this to get

\[ A' = 2 - \frac{288}{x^2}. \]

We set the derivative to zero and solve for our critical number...
The second derivative \( A'' = \frac{288}{x^3} > 0 \) for \( x = 12 \). Thus, the minimum paper will be used when the printed part of the page is \( x = 12 \) and \( y = \frac{72}{12} = 6 \). This makes the total page size 16 by 8.

**Reading Assignment**

Tan, sections 4.4 and 4.5

**Lesson Assignment**

*Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments" in the course syllabus. This lesson assignment is worth 100 possible points.*

Exercise 4.4: 2, 3, 5, 13, 18, 23, 35, 41, 49, 51
Exercise 4.5: 3, 4, 9, 12, 16, 19
Mid-Course Examination

Preparation

It is now time to prepare for and take the mid-course examination. If you are not going to take your exam at LSU-Baton Rouge, notify us of your proctor by sending the completed Exam Proctor Information Form located in the appendix of this course guide to the Independent & Distance Learning office.

Please read the College Examination Information instructions located in the appendix of this course guide for further details.

About the Mid-Course Examination

The mid-course exam is worth 34 percent of your final grade. Calculators are allowed on the exam, but students may not use programmable or graphing calculators. Three hours are allowed to complete the exam, which should be adequate time. The problems on the exam are very similar to the problems in the lesson assignments.
Lesson 10: Compound Interest

Key Terms

- simple interest
- present value
- continuous interest
- compound interest
- effective interest rate
- e
- future value

Lesson Introduction

Exponential and logarithmic functions play an important role in calculus. If you need to review exponential and logarithmic functions, you should study sections 5.1 and 5.2 of your text. We will discuss compound interest in this lesson.

If $P$ represents a principal of money, $r$ represents an interest rate per year, and $t$ represents time, then the simple interest on $P$ is given by the formula $I = Prt$. The accumulated principal and interest will be $A = P + I = P + Prt = P(1 + rt)$.

Example 1

If $1500 is invested for 8 years at 7% interest per year, then the accumulated principal and simple interest is $A = 1500\left(1 + (0.07)(8)\right) = 2340$.

Simple interest is normally not employed. Most of the time interest is paid on an account and then subsequent interest is paid on the original principal and the accumulated interest. This type of interest, interest paid on interest, is called compound interest.

Let a sum $P$ be invested at $r$ interest rate per year for $t$ years, and assume that interest is paid or “compounded” $n$ times per year. The amount, $A$, of accumulated compound interest is given by the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$
Lesson 10: Compound Interest

Example 2

Find the accumulated amount if $2000 is invested for 8 years at 6% annual interest with interest compounded semi-annually, quarterly, monthly, daily, and hourly!

semi-annual interest = \(2000 \left(1 + \frac{.06}{2}\right)^{2 \times 8} = 3209.41\)

quarterly interest = \(2000 \left(1 + \frac{.06}{4}\right)^{4 \times 8} = 3220.64\)

monthly interest = \(2000 \left(1 + \frac{.06}{12}\right)^{12 \times 8} = 3228.29\)

daily interest = \(2000 \left(1 + \frac{.06}{365}\right)^{365 \times 8} = 3232.02\)

hourly interest = \(2000 \left(1 + \frac{.06}{365 \times 24}\right)^{365 \times 24 \times 8} = 3232.14\)

Notice that as the frequency of compounding increases, the accumulated amount increases, but the amount of increase is decreasing. In the limiting case, as the number of compoundings tends to infinity, this formula will mutate into a different formula that we will discuss presently.

The effective interest rate is the simple interest rate that would lead to the same accumulated amount in one year as the nominal rate with compound interest. This leads to the following formula for effective interest rate.

\[ r_{\text{effective}} = \left(1 + \frac{r}{n}\right)^n - 1 \]

Example 3

A credit card advertises a monthly interest rate of 1.65% with interest calculated monthly. The nominal rate is 1.65% × 12 = 19.8%. The effective rate is higher, however. The effective rate is

\[ r_{\text{effective}} = \left(1 + \frac{.198}{12}\right)^{12} - 1 \approx .21699 \approx 21.70\% \]

A problem similar to the question of an accumulated amount is the calculation of the size of an initial deposit needed to grow to a specified amount. The
principal in a compound interest calculation is called the **present value**, and the accumulated amount is called the **future value**. Using this terminology, we can re-write the formula involving compound interest as

\[ P = A \left( 1 + \frac{r}{n} \right)^{nt} \]

### Example 4

A man needs to have $12,000 in 6 years to pay an obligation. How much should he invest now at 9% annual interest, compounded quarterly to have the $12,000 in 6 years? We calculate the present value of the $12,000 as follows.

\[
P = 12,000 \left( 1 + \frac{0.09}{4} \right)^{-4 \times 6} \approx 7,034.96\
\]

If the man deposits $7,034.96 in the account, it will grow to $12,000 with accumulated interest in 6 years.

One of the most important limits in mathematics is the calculation of

\[ \lim_{{x \to \infty}} \left( 1 + \frac{1}{x} \right)^x \]. This limit is discussed in Section 5.1 of your text. This limit is the number we call \( e \). The letter \( e \) was chosen to honor the great German mathematician Euler. The author discusses the calculation of this limit in table 5.1. Note that \( e \) is an irrational number that has an approximate decimal expansion of \( e \approx 2.718281828459 \).

Thus, we may write

\[ A = \lim_{{x \to \infty}} \left( 1 + \frac{1}{x} \right)^x = e \approx 2.718281828459. \]

Returning to the case of compound interest for a principal \( P \), with annual interest rate \( r \), for \( t \) years, compounded \( n \) times a year, let us examine the formula as the number of compoundings begins to increase. Specifically, we consider

\[ A = \lim_{{n \to \infty}} P \left( 1 + \frac{r}{n} \right)^{nt}. \]

Substituting \( \frac{r}{n} = \frac{1}{x} \), and \( n = nx \) will transform the above limit into

\[ A = \lim_{{x \to \infty}} P \left( 1 + \frac{1}{x} \right)^{x \cdot t} = \lim_{{x \to \infty}} P \left[ \left( 1 + \frac{1}{x} \right)^x \right]^{rt} = Pe^{rt}. \]
Lesson 10: Compound Interest

This type of compound interest, with an infinite number of compoundings, is called **continuous interest**.

Example 5

Take the figures from example 2 and compute the continuous interest. We had a $2000 principal for 8 years at a 6% annual interest rate. If we assume continuous compounding, the principal will grow in 8 years to

\[ A = 2000e^{0.06 \times 8} \approx 3232.15. \]

Notice the tiny additional increase in the amount over the hourly compounding result. Although it is true that more frequent compounding results in larger accumulations, these increases do get smaller and smaller and their limiting value is calculated using continuous interest.

Reading Assignment

Tan, section 5.3

Lesson Assignment

*Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.*

Exercise 5.3:  1, 4, 5, 7, 11, 17, 21, 23, 26, 27, 40
Lesson 11: Differentiation of Exponential Functions: Differentiation of Logarithmic Functions

Key Terms

\[ D_x \left( \ln h(x) \right) = \frac{1}{hx} \cdot h'(x) \quad D_x \left( \ln x \right) = \frac{1}{x} \]

\[ D_x \left( e^{kx} \right) = e^{kx} \cdot k(x) \quad D_x \left( e^x \right) = e^x \]

Lesson Introduction

We will now learn how to differentiate exponential and logarithmic functions.

**Theorem:** For \( x > 0 \), \( D_x \left( \ln x \right) = \frac{1}{x} \).

**Example 1**

With \( x > 0 \), we can easily compute \( y' \) for the function \( y = 3x^2 + 5 \ln x \). We differentiate and obtain \( y' = 6x + 5 \left( \frac{1}{x} \right) = 6x + \frac{5}{x} \).

**Example 2**

With \( y = (\ln x)^3 \), compute the equation of the tangent line to the graph of \( y \) at \( x = 4 \). We obtain, upon differentiating, \( y' = 3(\ln x)^2 \cdot \frac{1}{x} \). Thus, the slope of the tangent line at \( x = 4 \) is

\[ y'(4) = 3(\ln 4)^2 \cdot \frac{1}{4} = \frac{3}{4} \ln^2 (4). \]
With \( x = 4, (\ln 4)^3 \). Thus, the equation of the tangent line is

\[
y - (\ln 4)^3 = \frac{3}{4}(\ln^2 4)(x - 4).
\]

**Theorem:** If \( h(x) \) is a differentiable function of \( x \) and \( h(x) > 0 \), then

\[
D_x\left(\ln(h(x))\right) = \frac{1}{h(x)} \cdot h'(x).
\]

**Example 3**

\[
D_x\left(\ln(x^3 + 2x)\right) = \frac{1}{x^3 + 2x} \cdot (3x^2 + 2) \quad \text{and}
\]

\[
D_x\left(\ln(\ln x)\right) = \frac{1}{\ln x} \cdot \frac{1}{x}.
\]

**Theorem:** The derivative of \( e^x \) is \( e^x \), or \( D_x\left(e^x\right) = e^x \).

**Example 4**

Where is the derivative of \( y = e^x \) equal to \( 7 \) ? \( y' = (e^x)' = e^x = 7 \), which yields \( e^x = 7 \) or \( x = \ln 7 \).

**Theorem:** If \( h(x) \) is a differentiable function of \( x \), then we have

\[
D_x\left(e^{h(x)}\right) = e^{h(x)} \cdot h'(x).
\]

**Example 5**

If \( y = e^{x^3 + 4} \), then \( y' = e^{x^3 + 4} \cdot 3x^2 \) and

if \( y = e^{\sqrt{x}} \), then \( y' = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \).

**Example 6**

Let \( y = e^{7x^2 + 1} \). Write the equation of the tangent line to the graph of \( y \) when \( x = 1 \). We get, upon differentiating, \( y' = e^{7x^2 + 2} \cdot 14x \) and, thus, \( y'(1) = 14e^8 \). The equation of the desired line is \( y - e^8 = 14e^8(x - 1) \).
Lesson 11: Differentiation of Exponential Functions: Differentiation of Logarithmic Functions

**Reading Assignment**

Tan, sections 5.4 and 5.5

**Lesson Assignment**

*Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.*

Exercise 5.4:  1, 4, 9, 14, 17, 25, 34, 36, 43, 52
Exercise 5.5:  3, 8, 11, 15, 20, 27, 32, 36, 50, 56, 60, 63
Lesson 12: Exponential Functions as Mathematical Models

Key Terms

- exponential growth
- decay model
- half-life
- exponential decay
- growth constant
- growth model
- decay constant

Lesson Introduction

Often problems dealing that quantities that grow or shrink with respect to time can be modeled by exponential growth functions or exponential decay functions. Suppose $Q(t)$ is a quantity whose rate of growth is directly proportional to the present quantity $Q(t)$. Furthermore, suppose the initial quantity $Q(0)$ is denoted as $Q_0$. The mathematical formulation of these conditions can be written as $Q'(t) = kQ(t)$ and $Q(0) = Q_0$.

On page 380 your textbook, the author verifies that the function $Q(t) = Q_0e^{kt}$ satisfies the above conditions. More advanced texts will verify that this is the only function satisfying these conditions. If the quantity increases as time increases, $k$ is a positive number and we say $Q(t) = Q_0e^{kt}$ is a growth model. If the quantity decreases as time increases, $k$ is a negative number and we say $Q(t) = Q_0e^{kt}$ is a decay model. If $k$ is positive, it is called a growth constant, and if $k$ is negative, it is called a decay constant.

Example 1

Suppose a population of guppies obeys the exponential growth law. If we introduce 20 guppies to a large tank and the population is 100 guppies after 30 days, produce an exponential model for the population and use it to predict the population after 80 days and also determine when the population reaches 350.
Lesson 12: Exponential Functions as Mathematical Models

Because we start with 20 guppies, the initial equation can be viewed as $Q = 20e^{kt}$. We are given the information that after 30 days the population has grown to a hundred. Substitute $t = 30$ and $Q = 100$. Doing the necessary arithmetic gives us

$$100 = 20e^{30k} \iff 5 = e^{30k} \iff \ln 5 = 30k \iff \frac{\ln 5}{30} = k \approx 0.0536$$

So the equation for the guppy population now reads $Q(t) = 20e^{0.0536t}$. After 80 days, the population will be $Q(80) = 20e^{0.0536 \times 80} \approx 1456$. To determine when the population reaches 350, we must solve the equation $350 = 20e^{0.0536t}$ for $t$. The student is advised to verify that $t$ is approximately 53.4 days.

Radioactive substances decay according to the law of exponential decay. That is to say, the rate at which the radioactive substance decays is directly proportional to the amount of substance present.

Example 2

The half-life of a radioactive substances is 4.3 days. This means that the quantity of the substance decays by half after 4.3 days. Thus, if we begin with a 100-gram sample, only 50 grams will remain after 4.3 days. After another 4.3 days, only 25 grams will remain, and so forth. To the determine the decay constant $k$, we solve the equation $50 = 100e^{k \times 4.3}$ for the constant $k$. Simple arithmetic leads to $k \approx -0.1612$. We can now use this constant to write the general equation for the quantity present, $Q = 100e^{-0.1612t}$. After 12 days, our 100-gram sample has decayed to $100e^{-0.1612 \times 12}$, which approximates to 14.45 grams. Suppose we are interested in the problem in determining when the substance decays to 5 grams. Then we would solve the equation $5 = 100e^{-0.1612t}$ for the variable $t$. The student is advised to determine the answer is approximately 18.58 days.

A function of the type $Q(t) = C - Ae^{-kt}$, with all of the constants positive, can be used to model the way people and animals learn new topics. When $t = 0$, we see the value of $Q(t)$ is $C - A$. Notice that as $t$ tends to increase without bound, $e^{-kt}$ tends toward $0$. Thus, a horizontal asymptote for $Q(t)$ is $y = c$. 


The graph for $Q(t)$ is shown below.

![Graph of $Q(t)$]

This function $Q(t)$ and its associated graph have been shown to closely model the way people and animals learn. Knowledge is gained quickly at first, but the rate of learning, the derivative of $Q(t)$, decreases as time increases. In the long run, the rate of learning tends to slow to the point where new material is difficult to learn.

**Example 3**

A student is asked to memorize a set of random words, and suppose $Q(t) = 53 - 50e^{-0.01t}$ is a good model for the number of words a student has learned at time $t$ minutes. This asserts that the student begins the learning process with a knowledge of three words. $Q(120) = 53 - 50e^{-0.01\times120} \approx 38$, from which we infer that after two hours a student has learned 38 random words. In the long term (as time increases without bound), we see the student has the ability to memorize 53 random words. Suppose we are interested in the time required for the student to memorize 50 words. To answer this, we must solve the equation $50 = 53 - 50e^{-0.01t}$. The solution follows.

\[
\begin{align*}
50 &= 53 - 50e^{-0.01t} \\
-3 &= -50e^{-0.01t} \\
\frac{3}{50} &= e^{-0.01t} \\
\ln \left( \frac{3}{50} \right) &= -0.01t \\
\frac{\ln \left( \frac{3}{50} \right)}{-0.01} &= t \approx 281. 
\end{align*}
\]

281 minutes are needed for the student to learn 50 words.
Lesson 12: Exponential Functions as Mathematical Models

Reading Assignment

Tan, section 5.6

Lesson Assignment

Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.

Exercise 5.6: 1, 4, 7, 9, 13, 15, 16, 17
Lesson 13: Antiderivatives and the Rules of Integration

Key Terms

antiderivative
integration
indefinite integral
integrand

Lesson Introduction

We say that \( F(x) \) is an antiderivative of \( f(x) \) if \( F'(x) = f(x) \). Thus, an antiderivative of \( f(x) \) is just a function whose derivative is \( f(x) \).

Example 1

\[
\frac{1}{4}x^4 \text{ is an antiderivative of } x^3 \text{ since } D_x \left( \frac{1}{4}x^4 \right) = x^3.
\]

\[
e^x + 7 \text{ is an antiderivative of } e^x \text{ since } D_x \left( e^x + 7 \right) = e^x.
\]

\[
\left( \ln x \right) + 2 \text{ is an antiderivative of } \frac{1}{x} \text{ since } D_x \left( \left( \ln x \right) + 2 \right) = \frac{1}{x}.
\]

Notice that if \( F(x) \) is any antiderivative of \( f(x) \), then \( F(x) + c \) is also an antiderivative of \( f(x) \). We use the symbol \( \int f(x) \, dx \), called the indefinite integral, to denote all possible antiderivatives of \( f(x) \). The symbol \( \int \) is called the integral sign, the \( f(x) \) is called the integrand, and the \( dx \) is used to indicate that \( x \) is the variable with respect to which we wish to differentiate. The process of finding antiderivatives is called indefinite integration.

If \( F'(x) = f(x) \), then \( \int f(x) \, dx = F(x) + c \).
Lesson 13: Antiderivatives and the Rules of Integration

**Theorem:** \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \) if \( n \neq -1 \).

**Example 2**

\[
\int x^3 \, dx = \frac{x^4}{4} + c \\
\int \sqrt{x} \, dx = \int x^{\frac{1}{2}} \, dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c \\
\int \frac{1}{x^3} \, dx = \int x^{-5} \, dx = \frac{x^{-4}}{-4} + c
\]

**Theorem:**

\[
\int cf(x) \, dx = c \int f(x) \, dx \text{ where } c \text{ is a constant, and} \\
\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx.
\]

**Example 3**

\[
\int 3x^2 \, dx = 3 \int x^2 \, dx = 3 \frac{x^3}{3} + c = x^3 + c \\
\int (4x^5 + 2x) \, dx = 4 \int x^5 \, dx + 2 \int x \, dx = 4 \frac{x^6}{6} + 2 \frac{x^2}{2} + c = \frac{2}{3} x^6 + x^2 + c \\
\int \frac{5}{\sqrt{x}} \, dx = 5 \int \frac{1}{x^{\frac{1}{2}}} \, dx = 5 \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 10\sqrt{x} + c
\]

Notice that \( \int f(x) \, dx = F(x) + c \) is not a single function, but an infinite family of parallel functions. We say the family members are parallel because they all have the same derivative. If a point on a particular member of the family is given, we can determine the value of \( c \) and hence which member of the infinite family we seek.

**Example 4**

Determine \( F(x) \) such that \( F'(x) = 3x^3 + 2x + 1 \) and \( F(1) = 3 \). We have

\[
F(x) = \int (3x^3 + 2x + 1) \, dx = 3 \frac{x^4}{4} + 2 \frac{x^2}{2} + x + c = \frac{3}{4} x^4 + x^2 + x + c.
\]
Substituting, we get

\[ 3 = F(1) = \frac{3}{4} \left(1^4\right) + (1)^2 + 1 + c = \frac{11}{4} + c \Rightarrow 3 = \frac{11}{4} + c \Rightarrow c = -\frac{1}{4}. \]

Thus, the function in question is 

\[ F(x) = \frac{3}{4} x^4 + x^2 + x + \frac{1}{4}. \]

**Theorem:** \( \int \frac{1}{x} \, dx = \ln |x| + c \) and \( \int e^{kx} \, dx = \frac{1}{k} e^{kx} + c. \)

**Example 5**

\[
\int \frac{7}{x} \, dx = 7 \int \frac{1}{x} \, dx = 7 \ln |x| + c, \quad \int e^{4x} \, dx = \frac{1}{4} e^{4x} + c \quad \text{and} \quad \int e^{-x} \, dx = -3e^{-x} + c.
\]

**Example 6**

Evaluate \( \int e^{2x} \, dx \) if \( x = 0 \) implies \( F(x) = 2 \). We integrate to obtain \( \int e^{2x} \, dx = \frac{1}{2} e^{2x} + c. \)

Substituting gives us \( 2 = \frac{1}{2} e^{2(0)} + c \Rightarrow c = \frac{3}{2} \). Thus, \( F(x) = \frac{1}{2} e^{2x} + \frac{3}{2}. \)

**Example 7**

Suppose the marginal cost for making chairs is \( C'(x) = 20x - \sqrt{x} \) and the fixed cost is $4,000. Find the cost function. If \( C'(x) = 20x - \sqrt{x} \) then

\[ C(x) = \int \left(20x - \sqrt{x}\right) \, dx = 20 \frac{x^2}{2} - \frac{2}{3} x^{3/2} + c = 10x^2 - \frac{2}{3} x^{3/2} + c. \]
The fixed costs represent the cost when production is zero. Thus,

\[ 4000 = 10(0)^2 - \frac{2}{3}(0)^\frac{3}{2} + c \quad \text{or} \quad 4000 = c \]

The complete cost function is \( C(x) = 10x^2 - \frac{2}{3}x^\frac{3}{2} + 4000 \).

**Reading Assignment**

Tan, section 6.1

**Lesson Assignment**

*Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.*

Exercise 6.1: 1, 5, 8, 12, 17, 22, 27, 32, 35, 38, 45, 51, 54, 58, 60, 66
Lesson 14: Integration by Substitution

Key Term

differential

Lesson Introduction

Recall the chain rule gives us the ability to differentiate composite functions. In particular, the chain rule says,

\[ (f(g(x)))' = f'(g(x)) \cdot g'(x). \]

Example 1

Note the chain rule gives us \( D_x \left( (x^3 + 2)^4 \right) = 4(x^3 + 2)^3(3x^2) \). Thus,

\[ \int 4(x^3 + 2)^3 3x^2 \, dx = (x^3 + 2)^4 + c. \]

This last statement is apparent from the derivative we computed above, but without that initial step it is unlikely we would have been able to easily see the antiderivative.

If \( u = h(x) \) is a differentiable function of \( x \), then we define the differential of \( u \) to be \( du = h'(x) \, dx \). This will enable us to develop a method for integrating composite functions. Since \( D_x \left( g(h(x)) \right) = g'(h(x))h'(x) \), we have the corresponding integral formula

\[ \int g'(h(x))h'(x) \, dx = g(h(x)) + c. \]
Lesson 14: Integration by Substitution

If we let \( u = h(x) \) and let \( du = h'(x) \, dx \), then we can substitute these into the integral to give us

\[
\int g'(h(x))h'(x) \, dx = \int g'(u) \, du = g(u) + c.
\]

Example 2

If \( u = 3x^3 \) then \( du = 9x^2 \, dx \). If \( u = e^{x^8} \), then \( du = (e^{x^8} \cdot 2x) \, dx \). If \( u = \ln(x^3 + 2) \), then \( du = \left(\frac{1}{x^3 + 2} \cdot 3x^2\right) \, dx \).

Example 3

Evaluate \( \int (x^3 + 2x + 1)^4 (3x^2 + 2) \, dx \). We notice that \( (x^3 + 2x + 1) \) is raised to a power and that the derivative of \( (x^3 + 2x + 1) \) is present in the integrand. Let \( u = (x^3 + 2x + 1) \), and this gives us the differential of \( u \) as \( du = (3x^2 + 2) \, dx \). We substitute this into the original integral to get

\[
\int (x^3 + 2x + 1)^4 (3x^2 + 2) \, dx = \int u^4 \, du = \frac{u^5}{5} + c = \frac{1}{5} (x^3 + 2x + 1)^5 + c.
\]

Notice that the final answer must be given in terms of the original variable.

Example 4

Evaluate \( \int \left(1 + \frac{1}{x}\right)^2 \left(\frac{1}{x^2}\right) \, dx \). Notice that \( D_x \left(1 + \frac{1}{x}\right) = \frac{1}{x^2} \), so the derivative of \( \left(1 + \frac{1}{x}\right) \) is nearly present in the integrand. Let \( u = \left(1 + \frac{1}{x}\right) \); then \( du = \frac{-1}{x^2} \, dx \) or \( -du = \frac{1}{x^2} \, dx \). We can now substitute into the original integral to obtain

\[
\int \left(1 + \frac{1}{x}\right)^2 \left(\frac{1}{x^2}\right) \, dx = \int u^2 (-du) = -\int u^2 \, du = -\frac{u^3}{3} + c = -\frac{1}{3} \left(1 + \frac{1}{x}\right)^3 + c.
\]
**Example 5**

Evaluate $\int e^{(x^3+2)}(x^2) \, dx$. Here we will let $u = x^3 + 2$ and then $du = 3x^2 \, dx$ or $\frac{1}{3} du = x^2 \, dx$. Thus, the original integral becomes

$$\int e^{(x^3+2)}(x^2) \, dx = \int e^u \frac{1}{3} du = \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + c = \frac{1}{3} e^{(x^3+2)} + c.$$ 

**Example 6**

Evaluate $\int \frac{1}{x \ln x} \, dx$. Let $u = \ln x$, then $du = \frac{1}{x} \, dx$. Thus,

$$\int \frac{1}{x \ln x} \, dx = \int \frac{1}{u} \, du = \ln |u| + c = \ln |\ln x| + c.$$ 

**Example 7**

Evaluate $\int x\sqrt{x+1} \, dx$. Here the choice of $u$ is not so obvious, but observe what happens if we let $u = x + 1$. If $u = x + 1$ then $x = u - 1$ and $dx = du$. Thus, the original integral becomes

$$\int x\sqrt{x+1} \, dx = \int (u-1)\sqrt{u} \, du = \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) \, du = \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{5} (x+1)^{\frac{5}{2}} - \frac{2}{3} (x+1)^{\frac{3}{2}} + c.$$ 

Practice heavily on this material because integration by substitution gives many people a great deal of trouble.

**Reading Assignment**

Tan, section 6.2
Lesson Assignment

Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.

Exercise 6.2: 1, 4, 9, 12, 17, 24, 27, 34, 36, 39, 44, 49, 53, 61
Lesson 15: Area and the Definite Integral: The Fundamental Theorem of Calculus

Key Terms

- area
- fundamental theorem of calculus
- limits of integration
- definite integral

Lesson Introduction

Let \( y = f(x) \) be continuous and suppose \( f(x) \geq 0 \) on the closed interval \([a, b]\). Consider the sketch below.

We seek to devise a way to compute the area beneath \( f(x) \), above the \( x \)-axis and between \( x = a \) and \( x = b \). Toward this end, we divide the interval \([a, b]\) into \( n \) equal parts, each having width \( \Delta x = \frac{b - a}{n} \). The endpoints of these
Lesson 15: Area and the Definite Integral: The Fundamental Theorem of Calculus

intervals will be: \( x_0 = a; x_1 = a + \Delta x; x_2 = a + 2\Delta x \) and in general we have \( x_i = a + i\Delta x \). See the picture below.

Construct a rectangle with height given by \( f(x_0) \) on the first interval. Construct a rectangle with height given by \( f(x_1) \) on the second interval and in general construct a rectangle of height \( f(x_{i-1}) \) on the \( i \)th subinterval. Consider the following figure:

The area of the first rectangle is \( f(x_0)\Delta x \). The area of the second rectangle is \( f(x_1)\Delta x \). In general, the area of the \( i \)th rectangle is \( f(x_{i-1})\Delta x \). If we sum these areas we get

\[
S_n = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x,
\]

which serves as an approximation to the area we seek. The larger we choose \( n \), the closer we would expect the approximating rectangles to imitate the area we need. We define the area under \( y = f(x) \) over \( y = 0 \) and above the interval \([a, b]\) to be

\[
\text{Area} = \lim_{n \to \infty} S_n,
\]

providing the limit exists. When this limit exists we denote it by \( \int_a^b f(x)\,dx \), and call it the definite integral of \( f(x) \) over the interval \([a, b]\). In \( \int_a^b f(x)\,dx \), the \( b \) and the \( a \) are called the upper and lower limits of integration.
respectively. It is important to notice that \( \int f(x) \, dx \) represents an infinite family of functions, whereas \( \int_a^b f(x) \, dx \), is a number.

**Example 1**

What is the value of \( \int_0^1 (2x+1) \, dx \)? This definite integral represents the area beneath \( y = 2x + 1 \) over the interval \([0,1]\). The figure follows.

We can see the area of the lower portion of the figure is 1 and the area of the upper triangle is \( 1 \cdot \frac{1}{2} = \frac{1}{2} \). Thus, \( \int_0^1 2x + 1 = 2 \).

There must be a better way to evaluate a definite integral!

**The Fundamental Theorem of Calculus:** Let \( F(x) \) be any antiderivative for the continuous function \( f(x) \) on the interval \([a,b]\). Then,

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

**Example 2**

Use the Fundamental Theorem to evaluate the integral from example 1, \( \int_0^1 (2x+1) \, dx \). One antiderivative of \( 2x + 1 \) is \( F(x) = x^2 + x \). Thus,

\[
\int_0^1 (2x+1) \, dx = F(1) - F(0) = (1^2 + 1) - (0^2 + 0) = 2
\]

We use the symbol \( F(x) \int_a^b \) to indicate \( F(b) - F(a) \).
Lesson 15: Area and the Definite Integral: The Fundamental Theorem of Calculus

Example 3

Evaluate \( \int_2^4 (x^2 + 4) \, dx \). An antiderivative of \((x^2 + 4)\) is \( \frac{1}{3}x^3 + 4x \). Thus,

\[
\int_2^4 (x^2 + 4) \, dx = \left. \left( \frac{1}{3}x^3 + 4x \right) \right|_2^4 = \left( \frac{1}{3}4^3 + 4(4) \right) - \left( \frac{1}{3}2^3 + 4(2) \right) = \frac{64}{3} + 16 - \frac{8}{3} - 8 = \frac{56}{3} + 8 = \frac{80}{3}.
\]

Example 4

Evaluate \( \int_0^1 e^{5x} \, dx \). We know an antiderivative for \( e^{5x} \, dx = \frac{1}{5}e^{5x} \). Thus,

\[
\int_0^1 e^{5x} \, dx = \frac{1}{5} \int_0^1 e^{5x} \, dx = \frac{1}{5} \ln |x| \bigg|_0^1 = \frac{1}{5} (\ln 3 - \ln 1) = \frac{1}{5} \ln 3.
\]

Example 5

Evaluate \( \int_1^3 \left( \frac{x + 4}{x + 2} \right) \, dx \).

The antiderivative must be found by substitution, as in Lesson 14. It is possible to evaluate the definite integral entirely in terms of \( u \), and thus bypass the process of returning to the original variable of integration.

When we see \( \int_1^3 \left( \frac{x + 4}{x + 2} \right) \, dx \), it is implicit that the limits of integration are \( x = 1 \) and \( x = 3 \).

The limits of integration are based on \( x \). Choose \( u = x + 2 \) and thus \( x = u - 2 \), and lastly \( du = dx \). If \( x = 3 \), then \( u = x + 2 = 3 + 2 = 5 \) and if \( x = 1 \), then \( u = 3 \). So for the limits of integration \( x = 1 \) and \( x = 3 \), we now have \( u = 3 \) and \( u = 5 \). We substitute back into the original.

\[
\int_1^3 \left( \frac{x + 4}{x + 2} \right) \, dx = \int_3^5 \left( \frac{u - 2}{u} + 4 \right) \, du = \int_3^5 \left( 1 + \frac{2}{u} \right) \, du = \left( u + 2 \ln |u| \right) \bigg|_3^5 = (u + 2 \ln |u|) \bigg|_3^5 = (5 + 2 \ln 5) - (3 + 2 \ln 3) = 2 + 2 \ln \frac{5}{3}.
\]
Lesson 15: Area and the Definite Integral: The Fundamental Theorem of Calculus

In summary, if \( f(x) \) is a continuous function on the closed interval \([a, b]\) and assuming \( f(x) \) is greater than or equal to zero on the integral, then the value of the definite integral gives the area underneath \( f(x) \), above the \( x \)-axis and between \( x = a \) and \( x = b \). In the event \( f(x) \) assumes some negative values, the reader is urged to consider the geometric interpretation on page 427.

Reading Assignment

Tan, sections 6.3 and 6.4

Lesson Assignment

*Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.*

Exercise 6.3:  1, 3, 7, 13, 15  
Exercise 6.4:  1, 4, 7, 12, 15, 20, 25, 29, 34, 37, 41
Lesson 16: Evaluating Definite Integrals: Area between Two Curves

Key Terms

area  area between two curves
average value of a function

Lesson Introduction

The student is advised to study the list of properties for the definite integral that are listed on page 442 of the textbook. Note that some of these properties follow from the corresponding properties of indefinite integrals.

It is possible to use substitution to evaluate definite integrals. If you try to integrate a composition of two functions using a substitution, you can calculate the antiderivative, return to the original variable and then employ the Fundamental Theorem. Another method is to make the substitution $u = h(x)$ and then use this substitution to change the limits of integration from limits of $x$ to limits in terms of $u$. Both techniques are illustrated in the following example.

Example 1

Evaluate $\int_{1}^{2} x^3 \left( x^3 + 3 \right)^3 \, dx$. The first method will not involve changing the limits of integration.

We start by choosing $u = x^3 + 3$ and thus $du = 3x^2 \, dx$ or $\frac{1}{3} \, du = x^2 \, dx$.

Calculating an antiderivative for the integral gives us
Lesson 16: Evaluating Definite Integrals: Area between Two Curves

\[
\int_{x=1}^{x=2} x^2 (x^3 + 3)^3 \, dx = \int_{u=1}^{u=4} \frac{1}{3} u^3 du = \frac{1}{3} \left[ \frac{u^4}{4} \right]_{v=1}^{v=2} = \frac{1}{3} \left( \frac{(x^3 + 3)^4}{4} \right)_{v=2}^{v=1} = \frac{1}{12} (2^3 + 3)^4 - \frac{1}{12} (1^3 + 3)^4 = 1198.75.
\]

A second method of attack is to change the limits of integration to reflect the values of \( u \). We again choose \( u = x^3 + 3 \). If we substitute the limits of integration, \( x = 1 \) and \( x = 2 \), into the equation for \( u \), we get \( x = 4 \) and \( x = 11 \), respectively. We can use the values to work the integral in terms of \( u \) alone, not requiring a return to the original variable of integration. This approach is detailed below.

\[
\int_{x=1}^{x=2} x^2 (x^3 + 3)^3 \, dx = \int_{u=1}^{u=11} \frac{1}{3} u^3 du = \frac{1}{3} \left[ \frac{u^4}{4} \right]_{u=4}^{u=11} = \frac{1}{12} \left( (11^4 - 4^4) \right) = 1198.75.
\]

The student can employ either technique. Some people prefer to alter the limits of integration as this results in a slightly shorter solution.

As mentioned in the previous lesson, the definite integral can be used to evaluate the area beneath a curve. If \( f(x) \) is continuous and non-negative on the interval \([a, b]\) then the area beneath the function, above the \( x \)-axis and between \( x = a \) and \( x = b \) is given by the definite integral

\[
\int_a^b f(x) \, dx.
\]

Example 2

Find the area beneath \( y = f(x) = x^2 + 1 \), above the \( x \)-axis and between \( x = 1 \) and \( x = 2 \). The required area is sketched below.
Lesson 16: Evaluating Definite Integrals: Area between Two Curves

According to our previous discussions, the desired area is \( \int_{1}^{2} (x^2 + 1) \, dx \). The area is calculated as follows:

\[
\int_{1}^{2} (x^2 + 1) \, dx = \frac{x^3}{3} + x \bigg|_{1}^{2} = \left( \frac{8}{3} + 2 \right) - \left( \frac{1}{3} + 1 \right) = \frac{10}{3}.
\]

We define the **average value** of a continuous function on \([a, b]\) by using an integral. If \( f(x) \) is continuous on \([a, b]\) we define the **average value** of the function, written \( \overline{f}(x) \) to be

\[
\overline{f}(x) = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx.
\]

**Example 3**

What is the average value of the function \( f(x) = 3x^3 + 2x - 5 \) on the interval \([-1, 2]\) ? We calculate the desired quantity below.

\[
\overline{f}(x) = \frac{1}{2 - (-1)} \int_{-1}^{2} (3x^3 + 2x - 5) \, dx = \frac{1}{3} \left[ 3 \frac{x^4}{4} + x^2 - 5x \right]_{-1}^{2} = \frac{1}{3} \left( 6 - \frac{27}{4} \right) = \frac{-1}{4}
\]

If \( f(x) \geq g(x) \) for all \( x \) in the interval \([a, b]\), then the **area between the curves** \( y = f(x) \) and \( y = g(x) \) and between the lines \( x = a \) and \( x = b \) is given by the integral formula

\[
\text{Area between curves} = \int_{a}^{b} \left[ f(x) - g(x) \right] \, dx.
\]

A region of this type is sketched as follows.
Lesson 16: Evaluating Definite Integrals: Area between Two Curves

Example 4

Compute the area bounded by the curves \( y = x^2 \) and \( y = 2 - x \). The sketch of the region is shown here.

We can equate the y values of the two curves to determine the points of intersection. Thus,

\[
x^2 = 2 - x \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x + 2)(x - 1) = 0 \Rightarrow x = 1, -2.
\]

Since we can see from the graph \( 2 - x > x^2 \) on the interval \([-2, 1]\), we can obtain the area by an integral in the following manner.

\[
\text{Area} = \int_{-2}^{1} [(2 - x) - (x^2)] \, dx = \left[ 2x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-2}^{1}
\]

\[
= \left( 2 - \frac{1}{2} - \frac{1}{3} \right) - \left( -4 - 2 + \frac{8}{3} \right) = \frac{9}{2}
\]
Example 5

Calculate the area between the \( x \) axis and the curve \( y = e^x - 1 \) on the interval \([-1, 2]\). The sketch of the region is shown below.

![Graph](image)

Notice that the curve crosses the \( x \) axis at the origin. The \( x \) axis is the function \( y = 0 \). The \( x \) axis is the larger curve on the interval \([-1, 0]\) while the given function is larger on the interval \([0, 2]\). The area must be computed as two separate integrals. The area from \( x = -1 \) to \( x = 2 \) is given by the integral as

\[
\int_{-1}^{0} \left[ (0) - (e^x - 1) \right] \, dx = \left[ x - e^x \right]_{-1}^{0} = (0 - e^0) - (-1 - e^{-1}) = e^{-1}
\]

The area from \( x = 0 \) to \( x = 2 \) is given by

\[
\int_{0}^{2} \left[ (e^x - 1) - (0) \right] \, dx = \left[ e^x - x \right]_{0}^{2} = (e^2 - 2) - (0) = e^2 - 3
\]

The sum of the two calculations will give us the desired area. Thus, the area is \( e^{-1} + e^2 - 3 \).

Reading Assignment

Tan, sections 6.5 and 6.6
Lesson Assignment

Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.

Exercise 6.5:  1, 8, 12, 14, 17, 23, 31, 36, 44  
Exercise 6.6:  1, 4, 8, 11, 17, 23, 36, 39, 41
Lesson 17: Applications of the Definite Integral to Business and Economics

Key Terms

- supply $S(x)$
- equilibrium point
- annuity
- Gini index
- demand
- equilibrium quantity
- consumers’ surplus term $D(x)$
- equilibrium price
- producer’s surplus
- Lorentz curve

Lesson Introduction

In general, the price of an item determines the **supply** and the **demand** for the item. As price increases demand for the item usually falls. Conversely, as the price increases, the quantity producers are willing to supply will increase. We can view these relationships in a different light by letting the production variable, $x$, be the independent variable. Let $P_c = D(x)$ be the price at which consumers are willing to pay if there is a quantity of $x$ items produced. Let $P_s = S(x)$ be the price at which producers are willing to supply items with a production of $x$ items. $D(x)$ should be a decreasing function and $S(x)$ should be an increasing function. Consider the graph below.
The quantity where the demand and supply functions meet is called the **equilibrium quantity** (EQ), and the corresponding price is called the **equilibrium price** (EP). The actual point of intersection is called the equilibrium point. We assume that the market establishes itself in such a manner that the equilibrium point is achieved. There are consumers willing to pay more than the equilibrium price for the product. When the product sells at the equilibrium price, these consumers realize a savings. The difference between what these consumers would be willing to pay and what they have to pay is called the **consumers’ surplus**.

Geometrically, this represents the area shaded with horizontal lines above. This area is calculated with an integral as

\[
\text{Consumers’ surplus} = \int_{0}^{\text{EQ}} [D(x) - EP] \, dx.
\]

Similarly, we can talk about the producers’ concerns. At an established equilibrium price, there are producers who would be willing to supply the item at a lower price. The difference between what the producers earn and what they would have been willing to accept is called the **producers’ surplus**.

Geometrically, this represents the area shaded with slanted lines in the previous graph. This area is calculated with an integral as

\[
\text{Producers’ surplus} = \int_{0}^{\text{EQ}} [EP - S(x)] \, dx.
\]

**Example 1**

Suppose the supply curve is given by \( D(x) = 2 - \frac{x}{2} \) and let \( S(x) = \frac{x^2}{4} \). Setting the two functions equal to find the equilibrium points gives us
Lesson 17: Applications of the Definite Integral to Business and Economics

\[ 2 - \frac{x}{2} = \frac{x^2}{4} \implies x^2 + 2x - 8 = 0 \implies (x + 4)(x - 2) = 0. \]

We select the positive solution \( x = 2 \) and this forces \( y = 1 \). The consumers’ surplus is given by the following calculation.

Consumers’ surplus

\[
\left[ \int_0^2 \left( 2 - \frac{x}{2} \right) - 1 \right] dx = \left[ \int_0^2 \left( 1 - \frac{x}{2} \right) \right] dx
\]

\[
= \left( x = \frac{x^2}{4} \right) \bigg|_0^2 = 2 - 1 = 1.
\]

The producers’ surplus is calculated below.

Producers’ surplus

\[
\int_0^2 \left( 1 - \frac{x^2}{4} \right) dx = \int_0^2 \left( x - \frac{x^3}{12} \right) dx
\]

\[
= \left( 8 - \frac{8}{12} = \frac{4}{3} \right)
\]

A set of payments made at regular time intervals is called an \textbf{annuity}. The time during which payments are made is called the \textbf{term} of the annuity. Assume the payments are all equal in size. This type of arrangement is common in car loans, house loans, and insurance payments. Assume that we can earn continuous interest on these payments. If \( P \) is the payment, \( r \) is the continuous interest rate, \( t \) is the term in years, and \( m \) is the number of payments per year, then the future value of the annuity, \( A \), is derived with an integral on page 470 of your text to be

\[
\text{Future Value} = A = \frac{mP}{r} \left( e^{rt} - 1 \right).
\]

In a similar manner, we can derive the equation for the present value of the annuity as

\[
\text{Present Value} = PV = \frac{mP}{r} \left( 1 - e^{-rt} \right).
\]

\textbf{Example 2}

Determine the present and future values of an annuity of $300 every month for 20 years if the continuous interest rate is 6%. Using the above equations gives us a present value of

\[
PV = \frac{12 \times 300}{0.06} \left( 1 - e^{-0.06 \times 20} \right) \approx 41928.35.
\]
Lesson 17: Applications of the Definite Integral to Business and Economics

Using the formula to determine the future value of the payments will lead us to

\[ A = \frac{12 \times 300}{.06} \left( e^{.06 \times 20} - 1 \right) = 139207.02. \]

The future value is much larger than the present value. This is because of the 20 years worth of interest it will have accrued.

A Lorentz curve is a continuous, increasing, concave up curve, defined on the interval \([0,1]\), with a range equal to \([0,1]\) as well. Furthermore, a Lorentz curve passes through the origin and through the point \((1,1)\). A Lorentz curve can be used to describe what portion of a population has what portion of that population’s wealth. In particular, if \(f(x)\) is a Lorentz function then we define \(f(x) = (\text{the proportion of total income earned by the poorest } x\% \text{ of the population})\). In these terms, \(f(.60) = .48\) means the poorest 60% of the population earns only 48% of the total income. If \(f(x) = x\), we would see that the poorest \(n\%\) of the population would have \(n\%\) of the wealth. This corresponds to a socialist economy where all people have the same incomes. If we measure the deviation of the Lorentz curve, \(f(x)\), from the pure socialist model \(y = x\), we can see how far from this socialist state the economy will lie.

This measurement, called the **Gini index** of the Lorentz curve, \(f(x)\), is defined by the following integral.

\[ \text{Gini index} = 2 \int_0^1 [x - f(x)] \, dx. \]

If the Gini index were 0, this would correspond to a purely socialist distribution of wealth. As the index increases to its maximum value of 1, the income is seen to be concentrated into the hands of fewer people.

**Example 3**

Suppose the Lorentz curve for a society is given by \(f(x) = \frac{14}{15} x^2 + \frac{1}{15} x\). A quick calculation reveals \(f(.50) \approx .27\). Thus, the poorest half of the population controls approximately 27% of the society’s wealth. To compute the Gini index, we calculate the following integral.

\[ \text{Gini index} = 2 \int_0^1 \left[ x - \left( \frac{14}{15} x^2 + \frac{1}{15} x \right) \right] \, dx = 2 \left( \frac{14}{15} \frac{x^2}{2} - \frac{14}{15} \frac{x^3}{3} \right) \bigg|_0^1 \approx .311. \]
Lesson 17: Applications of the Definite Integral to Business and Economics

The Gini index of .31 can be construed to represent a 31% deviation from a uniform distribution of a society’s wealth.

Reading Assignment

Tan, section 6.7

Lesson Assignment

*Complete the following and submit to LSU Independent & Distance Learning for grading. Be sure to follow the guidelines under “Preparation of Lesson Assignments” in the course syllabus. This lesson assignment is worth 100 possible points.*

Exercise 6.7: 1, 4, 6, 11, 16, 18, 21, 24
Final Examination

Preparation

It is now time to prepare for and take the final examination

YOU MUST EARN A PASSING AVERAGE ON THE EXAMINATIONS IN ORDER TO PASS THE COURSE.

About the Final Examination

The final exam is worth 30 percent of your final grade. The focus of the final exam will be on the material covered from Lesson 10 to Lesson 17. Calculators are allowed on the exam, but students may not use programmable or graphing calculators. Three hours are allowed to complete the exam, which should be adequate time. The problems on the exam will be very similar to those in the lesson assignments.

This study guide summarizes the course and provides examples and insights into calculus. This guide was written to be a learning tool, and it should help as a review for the final exam. Your final exam problems are very much like the lesson problems. If you can do the lesson assignments, then you should be able to pass the exam. Review briefly, then take the exam while everything is fresh in your mind. Good luck.

Paul Wayne Britt
Transcript Information

After you have completed this course, your grade will be filed with the Office of the University Registrar. If a transcript is needed, it is your responsibility to make a request in writing to:

Office of the University Registrar  
Louisiana State University  
Thomas Boyd Hall  
Baton Rouge, LA  70803  
FAX: 225-578-5991

To the Student

Congratulations on finishing the lesson assignments for your course. We hope you will continue your education by taking another course with us.

Our current bulletin is available online at www.outreach.lsu.edu/idl. You can also receive a copy of our latest bulletin by calling 800-234-5046. We look forward to hearing from you!
Appendix A

Contents

✓ College Examination Information
✓ Exam Proctor Information Form
Please follow these regulations:

You will only be allowed to take your examination when the IDL office has received and accepted all the assigned lessons.

You must bring a picture I.D. to your examination.

For additional rules concerning exam procedures, please refer to the Syllabus and Exam sections of this course guide.

If you change an exam proctor or address, you must notify IDL immediately so your exams can be routed correctly.

If you will take your exam at LSU-Baton Rouge, refer to the information in Section A, below.

If you cannot take your exam at LSU-Baton Rouge, refer to the information in Section B.

SECTION A

Information for Students Taking Examinations at LSU-Baton Rouge

LSU IDL tests by appointment only. We offer one morning session and one afternoon session Monday through Friday and a morning session only on select Saturdays. Visit our Web site (www.outreach.lsu.edu/idl) to see which dates and times are available. Before scheduling your exam, make sure that you will be eligible to test by your selected date (see our Web site for eligibility requirements).
SECTION B

Information for Students Who Cannot Take Their Examinations at LSU-Baton Rouge

- Make arrangements with one of the following local officials to act as your testing supervisor:
  - College students → Testing center of an accredited college/university, college administrator or UCEA Correspondence Study Department
  - Overseas students → American University (school) or American Embassy
  - Military personnel → Education office at the military base, or college locations listed above

- You must submit your Exam Proctor Information using the form in the Appendix of this course guide, or if you have access to the Internet, you may submit this information through the LSU IDL Web site (www.outreach.lsu.edu/idl).

You need to submit only one proctor information form per course to the IDL office. Any subsequent exams you need to take for the same course will be sent to the same proctor.

The proctor information form should be submitted as soon as you have found a proctor and must be received by the IDL office at least three lessons before you are ready to take your exam. Receipt of this form by the IDL office does not mean your exam will be sent immediately. Your exam will be mailed to your proctor after the IDL office has received and accepted all lessons that must be completed prior to taking the exam.

Your exam proctor will hold your examination for no longer than thirty days. You should check to be certain the exam has arrived; if not, notify this office immediately. You must make arrangements for a time to take your exam, and you are responsible for any proctor fees.

If you change an exam proctor or address, please notify IDL immediately so your exams can be routed correctly.
Appendix A

Exam Proctor Information Form

Before you complete this form, please read the preceding examination information.

Directions

• If you will take your exam at LSU-Baton Rouge, you do not need to complete this form.
• Do not send this form with one of your lessons; send it separately to the IDL office.
• If you have any questions concerning this form, please call the IDL office at 225-578-2500 or 800-234-5046.
• If you have access to the Internet, you may submit this information through the LSU IDL Web site (www.outreach.lsu.edu/idl).

Enrollment Number ________________________________________________________________

Course Name ______________________________________________________________________

Student Name _____________________________________________________________________

Address __________________________________________________________________________

Telephone __________________________ E-mail ____________________________________

☐ Check the box if this is an address change from your original enrollment.

Complete the information below with reference to the person who will proctor your exam.

Exam Proctor’s Name _______________________________________________________________

Exam Proctor’s Title ________________________________________________________________

Office Telephone _________________________________________________________________

E-mail Address _________________________________________________________________

College/University _______________________________________________________________

Department/Section _______________________________________________________________

Building, Street, or P. O. Box _______________________________________________________

City, State, Zip Code _____________________________________________________________

✉ Mail to: LSU Independent & Distance Learning

1225 Pleasant Hall
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Baton Rouge, LA 70803
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